Application of Unscented Kalman Filter for Non-linear Estimation in Deformation Monitoring

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Abstract: The Extended Kalman Filter has been one of the most widely used methods for estimation of non-linear systems through linearizing non-linear models. In recent several decades people have realized that there are a lot of constraints in application of the EKF for its hard implementation and intractability. In this paper a new estimation method is proposed, which takes advantage of the Unscented Transformation method thus approximating the true mean and variance more accurately. The new method can be applied to non-linear systems without the linearization process necessary for the EKF, and it does not demand a Gaussian distribution of noise and what's more, its ease of implementation and more accurate estimation features enables it to demonstrate its good performance in the experiment of deformation monitoring. Numerical experiments show that the application of the Unscented

Kalman Filter in deformation prediction is more effective than that of the EKF.

Key words: EKF; Unscented Transform; Sigma Point Transform; deformation monitoring

1. Introduction

In practical problems of engineering, whenever the state of a system must be estimated from noisy sensor information, some kind of state estimator is made use of to fuse the data from different sensors together to produce an accurate estimate of the true system state. As is well known, the Kalman Filter(KF) is always used to deal with the system whose dynamics and observation models are linear, and the Extended Kalman Filter(EKF) is the most widely used estimator for nonlinear systems. The EKF applies the Kalman filter to nonlinear systems by simply linearizing all the nonlinear models so that the traditional Kalman filter equations can be applied. However, as people have found, the use of the EKF in practice has two well-known drawbacks. The first one is that linearization can produce highly unstable filters if the assumptions of local linearity is violated. The second one is that the derivation of the Jacobian matrices is nontrivial in most applications and often leads to significant implementation difficulties.

In this paper a new linear estimator is introduced which yields performance equivalent to the Kalman filter for linear systems, yet generalizes elegantly to nonlinear systems without the linearization steps required by the EKF. The new estimator uses the Unscented Transform proposed by Julier and Uhlmann（1996）to produce a set of points, which we name Sigma Points, and are then propagated to accurately approximate the true statistical properties of the random variables. As the word ‘Unscented’ is difficult to understand, thus we use ‘Sigma Point Transform’ and ‘Sigma Point Kalman Filter’ in this paper just for better understanding.

The structure of this paper is as follows. Section 2 introduces the Sigma Point transformation. Analysis is made to demonstrate this kind of transformation has better approximation performance. Section 3 introduces the Sigma Point Kalman Filter method. Section 4 presents an example. Conclusion is made within section 5. In the Appendix a detailed comparison is made to show that a higher accuracy of approximation of the statistical properties of the random variable after nonlinear propagation can be gained using the Sigma Point Transform.

2 Sigma Point Transformation

Sigma Point Transformation is in fact the Unscented Transformation proposed by Julier and Uhlmann（1996）to calculate statistical properties of random variables after nonlinear propagation. Consider problem: propagate a random variable x, whose dimension is d_x, through a nonlinear function $y = f(x)$. Assume x has mean value $\bar{x}$ and covariance P_x. To compute the statistical properties of y, the following formulas can be used

$$\begin{aligned} &\chi_0 = \bar{x} \\ &\chi_i = \bar{x} + \left(\sqrt{(d_x+\lambda)P_x}\right)_i \quad i = 1,\dots d_x \\ &\chi_i = \bar{x} - \left(\sqrt{(d_x+\lambda)P_x}\right)_{i-d_x} \quad i = d_x+1,\dots 2d_x \end{aligned} \tag{1}$$

in which χ is a matrix consisting of $2d_x+1$ vectors χ_i (called Sigma Point), $\lambda=\alpha^2\left(d_x+\kappa\right)-d_x$ being a scaling parameter, and constant α determines the extension of these vectors around $\bar{x}$ and usually set $1e-2\le\alpha\le1$[1]. κ is another scaling factor, and often set as zero for state estimation problems. $\left(\sqrt{\left(d_x+\lambda\right)P_x}\right)_i$ is the i-th column of the square root of the matrix. Here $\gamma=\sqrt{\left(d_x+\lambda\right)}$.

The acquired Sigma Points are transformed or propagated through the nonlinear function $f(\bullet)$

$$Y_i=f\left(\chi_i\right),\ i=0,1,\dots 2d_x \tag{2}$$

to obtain the transformed vectors Y_i. And then the mean value and covariance of y are approximated using the weighted mean and covariance of the transformed vectors

$$\bar{y}\approx\sum_{i=0}^{2d_x}\omega_i^{(m)}Y_i \tag{3}$$

$$P_y\approx\sum_{i=0}^{2d_x}\omega_i^{(c)}\left\{(Y_i-\bar{y})(Y_i-\bar{y})^T\right\} \tag{4}$$

in which weights ω_i are given as

$$\begin{aligned}\omega_0^{(m)}&=\lambda/\left(d_x+\lambda\right)\\ \omega_0^{(c)}&=\lambda/\left(d_x+\lambda\right)+\left(1-\alpha^2+\beta\right)\\ \omega_i^{(m)}&=\omega_i^{(c)}=1/\left\{2\left(d_x+\lambda\right)\right\}\quad i=1,2,\dots 2d_x\end{aligned} \tag{5}$$

And the parameter β contains the a priori information of x(for Gaussian Distribution, $\beta=2$)[2].

We can further prove that, using the Sigma Point Transformation, the statistical properties of the random variable after nonlinear transformation can be approximated with higher accuracy than that using simple linearization used in the Kalman Filter(The prove is shown in Appendix of this paper).

3 Sigma Point Kalman Filter Method

For the basic framework of the Extended Kalman Filter which involves the estimation of the state of a discrete-time nonlinear dynamic system

$$x_{k+1}=F\left(x_k,u_k,v_k\right) \tag{6}$$

$$y_k=H\left(x_k,\ n_k\right) \tag{7}$$

where x_k represents the unobserved state of the system and y_k is the only observed signal of the system. The outer input u_k is known and not a random variable. The process noise v_k

drives the dynamic system, and the observation noise is given by n_k. The system dynamic model F and H are assumed known. In state estimation, the EKF is the standard method of choice to achieve a recursive maximum likelihood estimation of the state x_k

$$\hat{x}_k = \left(predicted\ x_k \right) + K_k \cdot \left[y_k - \left(predicted\ y_k \right) \right] \tag{8}$$

The Sigma-Point Kalman Filter (SPKF) is a straightforward extension of the Sigma-Point Transform to the recursive estimation (8), where the random variable has been redefined as the concatenation of the original state and noise variables $x_k^a = \left[x_k^T \ v_k^T \ n_k^T \right]^T$. The sigma point selection scheme (1) is applied to this new augmented state vector to calculate the corresponding sigma matrix, χ_k^a. And the Sigma-Point Kalman Filter equations are given in Table 1. It must be pointed out that the implementation of the algorithm needs no explicit calculation of Jacobians or Hessians, which are always non-trivial burden of computation. And furthermore, the overall number of computations is the same order as that of the EKF[2].

As there often appears the special case where the process and measurement noise are additive, the computational complexity of the SPKF can be reduced. In such a case, the system state vector need not be augmented with the noise vector, which reduces the dimension of the sigma pints as well as the total number of sigma point used. The covariances of the noise sources are then incorporated into the state covariance using a simple additive procedure.

Table 1: Sigma Point Kalman Filter(zero mean noise case)

Initialization

$$\hat{x}_0 = E\left[x_0 \right]$$

$$P_0 = E\left[\left(x_0 - \hat{x}_0 \right)\left(x_0 - \hat{x}_0 \right)^T \right]$$

As for $k \in \{1, \cdots, \infty\}$

Compute Sigma Points:

$$\chi_{k-1} = \left[\hat{x}_{k-1} \ \ \hat{x}_{k-1} + \sqrt{\left(d_x + \lambda \right) P_{k-1}} \ \ \hat{x}_{k-1} - \sqrt{\left(d_x + \lambda \right) P_{k-1}} \right]$$

Prediction:

$$\chi_{k|k-1}^* = F\left[\chi_{k-1}, u_{k-1} \right]$$

$$\hat{x}_k^- = \sum_{i=0}^{2d_x} W_i^{(m)} \chi_{i,k|k-1}^*$$

$$P_k^- = \sum_{i=0}^{2d_x} W_i^{(c)} \left[\chi_{i,k|k-1}^* - \hat{x}_k^- \right]\left[\chi_{i,k|k-1}^* - \hat{x}_k^- \right]^T + R^v$$

$$\chi_{k|k-1} = \left[\hat{x}_k^- \ \ \hat{x}_k^- + \sqrt{\left(d_x + \lambda \right) P_k^-} \ \ \hat{x}_k^- - \sqrt{\left(d_x + \lambda \right) P_k^-} \right]$$

$$Y_{k|k-1} = H\left[\chi_{k|k-1} \right]$$

$$\hat{y}_k^- = \sum_{i=0}^{2d_x} W_i^{(m)} Y_{i,k|k-1}$$

Correction:

$$P_{\tilde{y}_k\tilde{y}_k} = \sum_{i=0}^{2d_x} W_i^{(c)} \left[Y_{i,k|k-1} - \hat{y}_k^- \right] \left[Y_{i,k|k-1} - \hat{y}_k^- \right]^T + R^n$$

$$P_{x_k y_k} = \sum_{i=0}^{2d_x} W_i^{(c)} \left[\chi_{i,k|k-1} - \hat{x}_k^- \right] \left[Y_{i,k|k-1} - \hat{y}_k^- \right]^T$$

$$K_k = P_{x_k y_k} P_{\tilde{y}_k\tilde{y}_k}^{-1}$$

$$\hat{x}_k = \hat{x}_k^- + K_k \left(y_k - \hat{y}_k^- \right)$$

$$P_k = P_k^- - K_k P_{\tilde{y}_k\tilde{y}_k} K_k^T$$

where λ is the composite scaling factor; d_x is the dimension of the state vector; R^v is the process noise covariance matrix; R^n is the measurement noise covariance matrix; W_i is weights as calculated in equation (5).

4. Numerical Experiment

For the predicting and filtering problem in deformation monitoring, we made numerical experiments based on descriptions in (Tor, 2002 and 2003). According to (Tor, 2003), Kalman filter is successful in weeding out the sudden surge in the readings even without incorporating a smoothing function and has robustness in avoiding the use of spurious observations. However, in our data processing, our main aim is to test what a performance improvement the SPKF method could gain compared to the EKF method when either system dynamic equation or observation equation is or both are nonlinear, so we just use simulated data of height displacements for the experiment, and we simplify the state vector as much as possible. The simulated data is a non-periodic and non-convergent time series acquired using a nonlinear auto regression model. By adding Gaussian white noise to this time series a noised observation series can be obtained

$$y_k = x_k + n_k \tag{10}$$

The state-space representation is the following state transition equation combined with (10)

$$x_k = f\left(x_{k-1}, v_{k-1} \right) \tag{11}$$

where $v_{k\text{-}1}$ is the model parameter. Note that in equation (11) there is no outer input.

In the estimation problem, the noisy time series y_k is the only observed input to the EKF and SPKF methods. Figure 1 and 2 shows a sub-segment of the estimates generated by the EKF and the SPKF respectively, and Figure 3 shows the difference of estimates between the EKF and the SPKF. The superior performance of the SPKF is clearly visible.

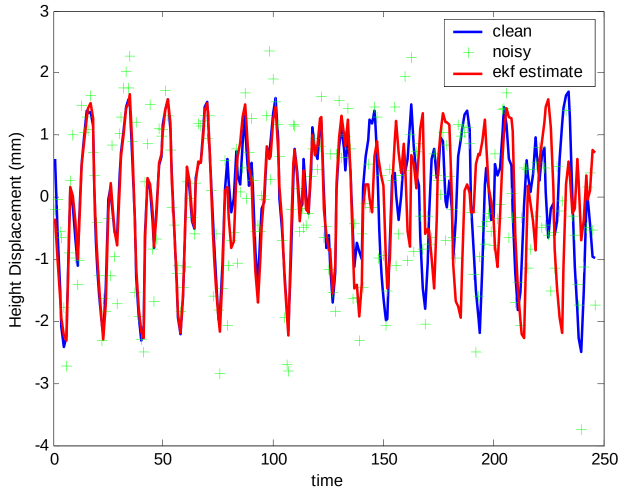

Figure 1: ekf : Mean square error (MSE) of estimate : 0.77069

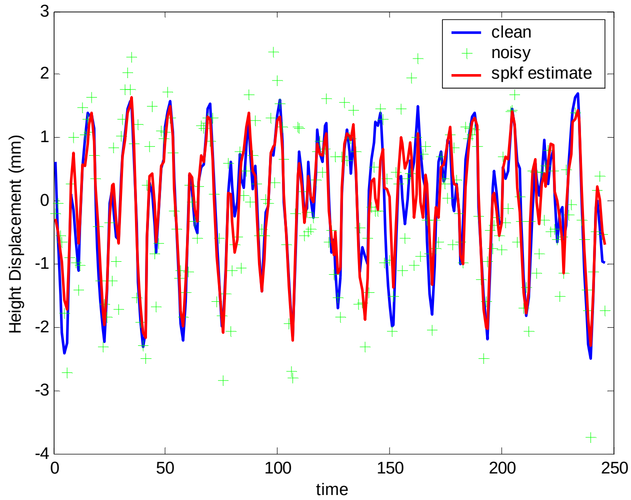

Figure 2: spkf : Mean square error (MSE) of estimate : 0.16437

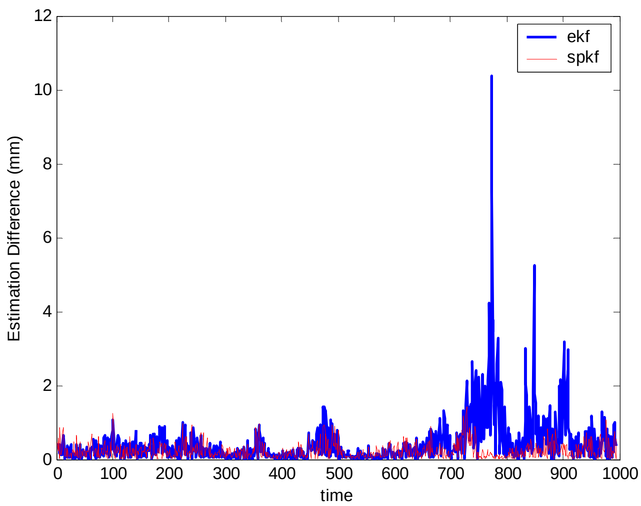

Figure 3: Estimation difference between ekf and spkf

5 Conclusion

The KF as well as the EKF have been the widely used linearity-based methods in a lot of engineering problems. However, in practical engineering there are always nonlinear situation where linearization could cause loss of accuracy. Therefore, this paper put forwards an alternative state estimation method to EKF which is preferable when dealing with nonlinear system model equations. The method is based on the Unscented Transform which uses a set of sigma points to approximate the actual statistical properties of random variable after nonlinear propagation with second order accuracy. Numerical results of the simulated deformation monitoring experiment show that the Sigma Point Kalman Filter method can gain great improvement in the accuracy of filtering. It is suggested that EKF can be replaced with the SPKF in nonlinear situation.

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Appendix: Accuracy Analysis

For the prediction of the future state or observation of a dynamic system, let us assume that x is a Gaussian random variable, with mean $\bar{x}$ and covariance P_x. Variable y is a nonlinear function of x

$$y = f(x) \tag{A1}$$

The Taylor series expansion of $f(x)$ at the mean $\bar{x}$ can be represented as follows

$$f(x) = f(\bar{x} + \delta x) = \sum_{n=0}^{\infty} \left[\frac{\left(\delta x \cdot \nabla_x\right)^n f(x)}{n!} \right]_{x=\bar{x}} \tag{A2}$$

where $\nabla_x f(x) = \dfrac{\partial f(x)}{\partial x}$, which is the Jacobian of partial derivatives of $f(x)$ with respect to x. Define the following operator

$$D_{\delta x}^n f \triangleq \left[\left(\delta x \cdot \nabla_x\right)^n f(x) \right]_{x=\bar{x}} \tag{A3}$$

The Taylor series expansion of the nonlinear function $y = f(x)$ can be written as follows

$$y = f(x) = f(\bar{x}) + D_{\delta x} f + \frac{1}{2} D_{\delta x}^2 f + \frac{1}{3!} D_{\delta x}^3 f + \frac{1}{4!} D_{\delta x}^4 f + \cdots \tag{A4}$$

where δx is a zero-mean Gaussian random variable with covariance P_x. From the statistical properties of δx, the series expression of the actual mean is obtained

$$\bar{y} = f(\bar{x}) + \frac{1}{2}\left[\left(\nabla^T P_x \nabla\right) f(x) \right]_{x=\bar{x}} + E\left[\frac{1}{4!} D_{\delta x}^4 f + \frac{1}{6!} D_{\delta x}^6 f + \ldots \right] \tag{A5}$$

Considering the statistical properties of δx we can acquire the series expansion of the actual covariance after nonlinear transform or propagation

$$P_y = A_x P_x A_x^T - \frac{1}{4}\left\{\left[\left(\nabla^T P_x \nabla\right) f(x)\right]\left[\left(\nabla^T P_x \nabla\right) f(x)\right]^T\right\}_{x=\bar{x}}$$
$$+ E\underbrace{\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i!\,j!}D_{\delta x}^{i} f\left(D_{\delta x}^{j} f\right)^T\right]}_{i\neq j=1} - \underbrace{\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(2i)!(2j)!}E\left[D_{\delta x}^{2i} f\right]E\left[D_{\delta x}^{2j} f\right]^T\right]}_{i\neq j=1} \quad \text{(A6)}$$

in which A_x is the Jacobian of $f(x)$ at $\bar{x}$. According to the series expansion of the actual posterior mean and covariance, the linearization process used in the EKF can be easily obtained

$$\bar{y}_{LIN} = f(\bar{x}) \quad \text{(A7)}$$

$$\left(P_y\right)_{LIN} = A_x P_x A_x^T \quad \text{(A8)}$$

The comparisons between (5) and (7), (6) and (8) tell us that only when the second and higher order terms within the mean and the fourth and higher order terms within the covariance are really negligible, can the results through linearization be accurate. Actually, most systems are nonlinear. Linearization not only results in poor accuracy but needs to compute the Jacobians.

On the other hand, as the Sigma Points are calculated in the following way

$$\begin{aligned}\chi_i &= \bar{x} \pm \sqrt{(d_x+\lambda)}\sigma_i \\ &= \bar{x} \pm \tilde{\sigma}_i\end{aligned} \quad \text{(A9)}$$

where σ_i represents the i-th column of the square root of the covariance matrix P_x, which means $\sum_{i=1}^{d_x}(\sigma_i\sigma_i^T) = P_x$. Therefore, the nonlinear function can be expanded at $\bar{x}$

$$\begin{aligned}Y_i &= f(\chi_i) \\ &= f(\bar{x}) + D_{\tilde{\sigma}_i} f + \frac{1}{2}D_{\tilde{\sigma}_i}^2 f + \frac{1}{3!}D_{\tilde{\sigma}_i}^3 f + \frac{1}{4!}D_{\tilde{\sigma}_i}^4 f + \ldots\end{aligned} \quad \text{(A10)}$$

$$\bar{y}_{SP} = f(\bar{x}) + \frac{1}{2}\left[\left(\nabla^T P_x \nabla\right) f(x)\right]_{x=\bar{x}} + \frac{1}{2(d_x+\lambda)}\sum_{i=1}^{2d_x}\left(\frac{1}{4!}D_{\tilde{\sigma}_i}^4 f + \frac{1}{6!}D_{\tilde{\sigma}_i}^6 f + \ldots\right) \quad \text{(A11)}$$

Comparing (A11) with the series expansion of the actual mean, it is clear that the difference between the mean after Sigma Point Transform and that of the actual mean just lies in the third and the following terms.

Similarly, the expression of the posterior covariance after the Sigma Point Transform can be written

$$\left(P_y\right)_{SP} = A_x P_x A_x^T - \frac{1}{4}\left\{\left[\left(\nabla^T P_x \nabla\right) f\left(x\right)\right]\left[\left(\nabla^T P_x \nabla\right) f\left(x\right)\right]^T\right\}_{x=\bar{x}} +$$

$$\frac{1}{2\left(d_x+\lambda\right)}\sum_{k=1}^{2d_x}\left(\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i!\,j!}D_{\tilde{\sigma}_k}^{i} f\left(D_{\tilde{\sigma}_k}^{j} f\right)^T}_{i\neq j=1}\right) - \tag{A12}$$

$$\left(\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(2i)!(2j)!4\left(d_x+\lambda\right)^2}\sum_{k=1}^{2d_x}\sum_{m=1}^{2d_x}D_{\tilde{\sigma}_k}^{2i} f\left(D_{\tilde{\sigma}_m}^{2j} f\right)^T}_{i\neq j=1}\right)$$

A comparison with the expression of the actual covariance shows that there exists a better coincidence between the acquired covariance through Sigma Point Transform and that of the actual covariance, with the difference included in the fourth and higher terms.

From the above description it can be concluded that the Sigma Point Transform provides a better approximation to the true statistical properties.

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