13th FIG Symposium on Deformation Measurement and Analysis
4th IAG Symposium on Geodesy for Geotechnical and Structural Engineering

LNEC, LISBON 2008 May 12-15

THE CONCEPT OF NETWORK ROBUSTNESS BASED ON STRAIN ANALOGY AS SEEN IN THE LIGHT OF CONTINUUM MECHANICS

Wiktor GAMBIN [1], Zenon PARZYŃSKI [2], Witold PRÓSZYŃSKI [2]

[1] *Faculty of Mechatronics,*
[2] *Faculty of Geodesy and Cartography*
Warsaw University of Technology

Abstract. The Vaniček concept of network robustness to observation gross errors, being a merger of the reliability and strain analysis, is investigated with respect to basic assumptions and the resulting requirements of continuum mechanics. The main objective of the study is to precisely determine to what extent the strain analogy can be applied to robustness analysis of geodetic networks and to provide explanations throwing some more light on this analogy. The following aspects are considered:

- structure of a network and nature of geodetic observations as non-material links between the network nodes;
- the influence of the reference conditions upon propagation of the effects of observation gross error through a network (free networks, tied-up networks);
- methods of finding the elements of the strain tensor;
- practical usefulness of the robustness indices in evaluation of the network's quality.

The theoretical considerations are illustrated with simple numerical examples.

1. INTRODUCTION

This paper can be considered as a preliminary contribution to the work of the Task Force 6.1.7. "Continuum mechanics as a support for deformation monitoring, analysis and interpretation", established within the Working Group 6.1. of the FIG Commission 6. Among the objectives of the Task Force there is "adaptation of strain theory based on continuum mechanics to robustness analysis of geodetic networks"
Accordingly, the Vaniček concept of network robustness (Vaniček et al. 2001), being a merger of the reliability and strain analysis, is investigated with respect to basic assumptions and the resulting requirements of continuum mechanics. The main objective of the study is to determine to what extent the strain analogy can be applied to robustness analysis of geodetic networks and to provide explanations throwing some more light on this analogy. It should be noted that the robustness analysis proposed in (Vaniček et al. 2001) is the extension of using strain to strength analyses of networks (eg. Dare 1983, Dare and Vaniček 1983).
We base our approach upon the conviction that for using any type of physical analogy to network analysis and design it is necessary, prior to creating any tools for the purpose of the analysis, to define the scope of the analogy and indicate its possible limitations. From the appropriately chosen analogy we may expect disclosing some new properties of a network and getting new measures that describe network's behaviour. We may also expect to get the

possibility for formulating the procedures and criteria that enable one to shape the network's structure with respect to accuracy and robustness.

We shall focus our attention entirely on the responses of geodetic networks to undetectable observation errors, the networks being considered as analytical systems i.e. where no physical movements of the network points take place. A more complex problem arises (Michel and Person 2003) when these robustness properties are analyzed together with the physical behaviour of a network established on a deforming body (being a material continuum) in order to monitor its deformations.

2. REVISITING THE MECHANICAL STRENGTH ANALOGY FOR GEODETIC NETWORKS

For the purpose of the present paper we shall emphasize that geodetic network is a set of material points (nodes) whose relative positions are determined by observations of geometrical quantities, i.e. distances, angles, being non-material links between the nodes. Due to such a structure a network displays specific behaviour. When correcting the initial (i.e. approximate) coordinates of the nodes on basis of the network adjustment, and thus obtaining their final coordinates, the node marks on the ground do not change their physical positions. So, a network represented by the adjustment model is a purely analytical system and should be analysed as such.

The formulas that describe geometrical behaviour of such systems due to observation errors are analogical to those describing the behaviour of discrete, statically indeterminate mechanical systems (in general - trusses with rods connecting arbitrary nodes, or frames with elastic joints in the nodes). With the former, final geometry is obtained by the least squares (LS) principle, whereas with the latter the static equilibrium is reached by the principle of minimum total potential energy. Both the principles are formally identical and, as will be shown below, mutually interpretable expressions.

Let us consider a linear adjustment model with datum constraints. Minimizing the LS objective function $\Phi = \mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v}$, where $\mathbf{v} = \mathbf{A} \cdot \mathrm{d}\mathbf{X} - (\mathbf{l}^{\mathrm{obs}} - \mathbf{l}^{\mathrm{o}})$, and adding the constraints, we get the constrained normal equations in the form

$$\mathbf{A}^{T}\mathbf{P}\mathbf{A} \cdot d\mathbf{X} = \mathbf{A}^{T}\mathbf{P}(\mathbf{l}^{obs} - \mathbf{l}^{o}) \qquad (1a)$$

$$\mathbf{B} \cdot d\mathbf{X} = \mathbf{0} \qquad (1b)$$

where: **A** (n×u) – the design matrix (rank deficient), d**X**(u×1) – the vector of coordinate corrections to be determined, **P**(n×n) – the weight matrix (diagonal), $\mathbf{l}^{obs}, \mathbf{l}^{o}$ - the (n×1) vectors of the observed and the approximate values of measured quantities; **B** (w×u) – the coefficient matrix in datum constraints (of full rank), $\mathrm{w} \geq \mathrm{d}$, where d = u – rank **A**, and also rank $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = u$.

Denoting $\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} = \mathbf{K}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{P}(\mathbf{l}^{\mathrm{obs}} - \mathbf{l}^{\mathrm{o}}) = \mathbf{f}$ we shall rewrite (1) in the form

$$\mathbf{K} \cdot d\mathbf{X} = \mathbf{f} \qquad (2a)$$

$$\mathbf{B} \cdot d\mathbf{X} = \mathbf{0} \qquad (2b)$$

where (2a) corresponds to fundamental equation of mechanics for a system in equilibrium, and (2b) – the boundary conditions.

Therefore, the network adjustment problem can be interpreted as follows:
- we have got a discrete, statically indeterminate mechanical system with the stiffness matrix **K**. The system is loaded with the forces **f**, being the reactions on the imposed changes of relative positions of some nodes. We assume that the magnitudes of the forces **f** are such that the responses of the system are within the linear elasticity area. The task is to find the movements $\mathrm{d}\hat{\mathbf{X}}$ of all the nodes, assuming the boundary conditions $\mathbf{B} \cdot d\hat{\mathbf{X}} = \mathbf{0}$ and a behaviour of the system according to the principle of minimum of total potential energy.

The forces **f** reacting on the nodes come from the internal forces $\mathbf{P}(\mathbf{l}^{obs} - \mathbf{l}^{o})$ in the system elements, due to the required changes $(\mathbf{l}^{obs} - \mathbf{l}^{o})$ in their sizes. The i –th element of the matrix **P** is a stiffness parameter, based on the elasticity modulus of the corresponding system element. A network with distances only corresponds to a truss with hinged nodes, whereas the analogy for a network with distances and angles could be a frame with elastic joints in the nodes. For a change in a distance, the Young modulus of a rod is responsible, whereas a change in an angle is determined by the Kirchoff modulus for a pair of rods connected by elastic joint. The values $\hat{\mathbf{v}} = \mathbf{A} \cdot d\hat{\mathbf{X}} - (\mathbf{l}^{obs} - \mathbf{l}^{o})$ are the resulting changes in the sizes of all the elements of the system, necessary to obtain the state of equilibrium of the system.

The solution vector $\mathrm{d}\hat{\mathbf{X}}$ can be found, equivalently, from

$$d\hat{\mathbf{X}} = (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-}_{\mathbf{B}} \mathbf{A}^T \mathbf{P}(\mathbf{l}^{obs} - \mathbf{l}^{o}) \tag{3}$$

$$d\hat{\mathbf{X}} = (\mathbf{A}^T \mathbf{P} \mathbf{A} + c\mathbf{B}^T \mathbf{B})^{-1} \mathbf{A}^T \mathbf{P}(\mathbf{l}^{obs} - \mathbf{l}^{o}) \tag{4}$$

where: $(\mathbf{A}^T \mathbf{P} \mathbf{A})^{-}_{\mathbf{B}}$ - the matrix obtained by inverting the coefficient matrix in the extended normal equations. c – the scalar of properly chosen magnitude.

Using this mechanical strength analogy, without resorting to the concept of strain tensor, one may analyse behaviour of geodetic networks as if they were discrete mechanical systems.

3. ON APPLICABILITY OF STRAIN ANALOGY TO ROBUSTNESS ANALYSIS OF GEODETIC NETWORKS

The problem of applicability of strain tensor analysis to evaluation of network robust-ness (and more generally – network strength) is far more complex. A geodetic network, represented as an analytical model, does not constitute a continuum by its structure. To use strain tensor analysis we have to assume that the model responses to observation errors as if it were a continuum and that it yields a continuous point displacement field, expressed by algebraic functions of at least C^2 class. We shall denote this hypothetic field by

$$F\{ d\hat{\mathbf{X}}(\Delta l_k)\} = \begin{bmatrix} u = u(x,y) \\ v = v(x,y) \end{bmatrix} \tag{5}$$

where $d\hat{\mathbf{X}}$ is obtained from (3) or (4), with $\Delta\mathbf{l}$ as in (Vaniček et al. 2001) i.e. $(\Delta\mathbf{l})^T = [0 \ldots 0 \;\; \Delta_k \;\; 0 \ldots 0]^T$, Δ_k being a maximum undetectable error (MUE) in the k–th observation.

Theoretically, without specifying the equivalent mechanical system, being the material continuum, it is perhaps not possible to provide a convincing support for the above mentioned assumption, and hence, for the applicability of strain tensor analogy to network analysis.
The complexity of the problem lies, to a high degree, in the fact that we undertake the reverse task to that of discretization of a material continuum. In the latter, when looking for a suitable FEM mesh we are free to gradually increase its density until the values of computed parameters are stable enough. In the case of a geodetic network (analogous to a discrete mechanical system) we have a fixed structure of nodes and their links. Adding any additional link is out of the question, as this would yield a different network.
By necessity, we shall only confine ourselves to a statement that for any vector of node displacements obtained from either (3) or (4) we can find, by means of approximation, the continuous point displacement field, wherefrom by differentiation we may deter-mine for each node the elements of the tensor T called the displacement gradient (decomposed into the strain tensor and the tensor of local rotation), and finally the robustness indices ρ, ϖ, γ, i.e.

$$\rho = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^{1/2}; \quad \varpi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right); \quad \gamma = \frac{1}{2}\left\{\frac{1}{4}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right\}^{1/2} \tag{6}$$

where: ρ - mean strain (deformation in scale), ω – differential rotation, γ - total shear (deformation of local configuration).

This method of evaluation of robustness indices can be described as follows

$$d\hat{\mathbf{X}}(\Delta l_k) \xrightarrow{\textit{approximation}} F\{d\hat{\mathbf{X}}(\Delta l_k)\} \xrightarrow{\textit{differentiation}} T_i \rightarrow (\rho, \varpi, \gamma)_i \tag{7}$$

It should be noted that the process of approximation, where various types of approximating functions can be used, has a limited level of accuracy, which affects the accuracy of the determined values of the tensor T elements, and eventually the accuracy of the robustness indices. In fact, we cannot estimate this accuracy properly as we do not know the actual shape of the displacement field.
The evaluation of the robustness indices can also be done by using the difference method based on linear deformation models (as in Vaniček et al. 2001, but slightly modified)

$$d\hat{\mathbf{X}}(\Delta l_k) \xrightarrow{\textit{use of}\ \text{linear deformation models}} T_i \rightarrow (\rho, \varpi, \gamma)_i \tag{8}$$

In this method, for each network node we shall form the redundant system of linear equations to be solved for tensor T elements by the LS method

$$\begin{aligned} \Delta u_{ij} &= \left(\frac{\partial u_i}{\partial x}\right)(x_j - x_i) + \left(\frac{\partial u_i}{\partial y}\right)(y_j - y_i) \\ \Delta v_{ij} &= \left(\frac{\partial v_i}{\partial x}\right)(x_j - x_i) + \left(\frac{\partial v_i}{\partial y}\right)(y_j - y_i) \qquad j = 1,2, \ldots, r;\; j \neq i \end{aligned} \tag{9}$$

where i – denotes the node of interest, j – other node, $\Delta u_{ij} = u_j - u_i$, $\Delta v_{ij} = v_j - v_i$.

As the j –th nodes (in a number of r) we may choose either all the remaining nodes or the neighbouring nodes, i.e. those linked with the i – th node by direct observations. The tensor elements can also be computed for r = 2.

In matrix notation the system (9) and its solution will have the form

$$\mathbf{Bt} = \Delta\mathbf{u}\,; \qquad \hat{\mathbf{t}}_{LS} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\Delta\mathbf{u} \tag{10}$$

where: **B** (2r×4) – the coefficient matrix of full rank, **t** (4×1) **–** the vector of tensor elements, $\Delta\mathbf{u}$ (2r×1) – the vector of inter-node displacements.

With the use of the two methods numerical tests have been carried out (see Section 4) to investigate to what degree the system (10), as a linear deformation model, corresponds with the given inter-node displacements. And, consequently, how representative are the determined robustness indices for the deformations around a node (i.e. within the sectors formed by inter-node lines). It was also the objective of the tests to observe whether the above properties are correlated with internal reliability of the network.

3. THE DATUM EFFECT UPON THE SHAPE OF THE DISPLACEMENT FIELD

We shall first recall the notion of a datum, by indicating its three components:

Datum = {**a reference base**; **reference conditions**; **a coordinate system**},

where a coordinate system is an auxiliary component.
Hence generally, change of datum can be the change in any one, in any two or in all of these components.

Maintaining the notation for reference conditions as in Eq.(1b), i.e. $\mathbf{B}\cdot \mathrm{d}\mathbf{X} = \mathbf{0}$, we shall distinguish their two characteristic types:

* *non-distorting conditions*: w = d, i.e. with **B**(d×u) such, that the changes in network geometry due to observation errors will not be affected by the reference conditions (e.g. free networks);

* *distorting conditions*: w > d, i.e. with **B**(w×u), such that the changes in network geometry due to observation errors will be affected by the reference conditions (e.g. tied-up networks).

It is known that the reference base and reference conditions used for a network, influence the way the effects of observation gross error (or errors) are propagated through a network and hence, yield a resulting pattern of node displacements and consequently, the corresponding shape of the displacement field. .
Within the class of non-distorting conditions the shape of the displacement field is dependent only on geometrical structure of a network, and consequently, is invariant to the changes of the datum. Thus, the robustness analysis with the use of the non-distorting conditions discloses the properties of a network itself, not distorted by the datum. Although from the practical point of view the use of the distorting conditions is often necessary.

4. NUMERICAL TESTS

Figure 1 shows the network variants used in the test computations. The variants are ranked in an increasing number of observations, the nodes being kept the same. Thus, each successive network has higher internal reliability.

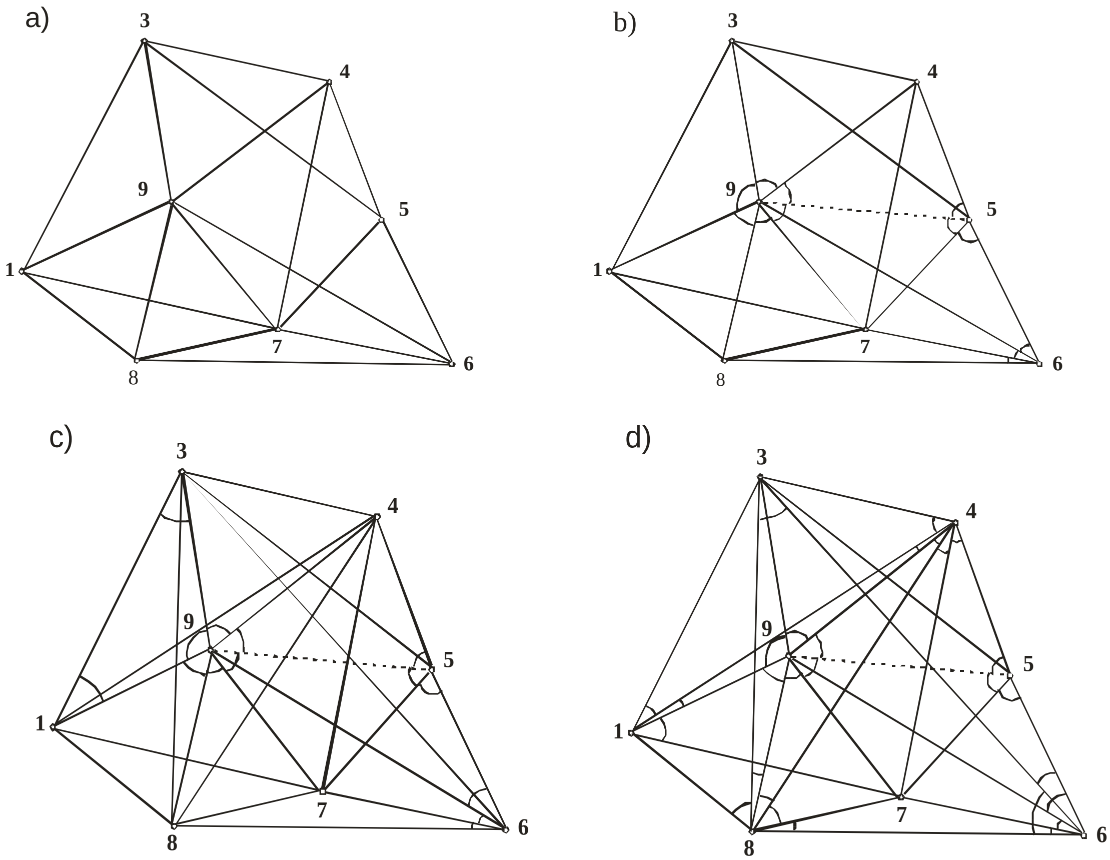

Fig. 1 - Network variants used in the tests

The displacements of the network nodes were generated by the maximum undetectable error (MUE) in the distance 4-7. The computations of robustness indices for the node 8 in each network variant have been done with the use of the difference method and the approximation. To examine the shape of the displacement field around this node the difference method has been also applied to pairs of the neighbouring lines linking this node with other nodes.

Links taken for a model	Net. variant 1 r [0.09 ÷ 0.66] MUE ≈ 10.2mm			Net. variant 2 r [0.28 ÷ 0.82] MUE ≈ 6.3mm			Net. Variant 3 r [0.34 ÷ 0.83] MUE ≈ 6.1mm			Net. variant. 4 r [0.44 ÷ 0.89] MUE ≈ 5.8mm		
	ρ	ω	γ	ρ	ω	γ	ρ	ω	γ	ρ	ω	γ
1, 9	1.06	10.15	2.16	0.58	0.46	0.55	-0.61	0.16	0.29	0.26	0.00	0.32
9, 7	2.89	22.14	4.31	1.38	1.79	0.70	0.51	2.02	0.98	0.54	0.89	0.54
7, 6	-66.48	-3.77	34.86	-3.74	-0.31	2.40	-5.89	-0.08	3.27	-2.39	-1.24	1.60
1, 6, 7, 9	0.69	8.79	2.82	0.99	0.93	0.13	-0.15	0.96	0.30	0.45	0.29	0.12
1,3,6,7,9, 4,5	1.87	1.78	1.74	1.55	0.30	0.25	0.76	0.34	0.12	0.91	0.10	0.21
	q = 1. 98		q* = 1.22	q = 0.40		q* = 0.40	q = 0.35		q* = 0.34	q = 0.18		q* = 0.17

Table 1- Robustness indices (in ppm) for node 8

To find out to what degree the linear deformation model based on the nodes 1,3,6,7,9,4,5 fits their displacements with respect to node 8, we shall use the index q for system inconsistency, defined as

$$q = \sqrt{\frac{\mathbf{v}^T\mathbf{v}}{2r-4}} \quad , \qquad \text{where:} \quad \mathbf{v} = \mathbf{B}\hat{\mathbf{t}}_{LS} - \Delta\mathbf{u}\,; \quad r = 7$$

q_* - the value of q reduced to common value of MUE, i.e. 6.3mm

Table 1 shows a big dispersion of the values of robustness indices among the model options for the weakest (in terms of internal reliability) network variant, gradually decreasing for successive variants of higher internal reliability. This tendency is also reflected in the decreasing values of the index q and its reduced form q_*. This indicates that the displacement field is most irregular for the weakest variant.

The similar trends in the results were observed for the node 9, where the discrepancies in the individual columns were smaller.

For comparison the method of polynomial approximation was applied to the set of all the network nodes. It yielded the values of robustness indices much smaller and less diversified than the corresponding values for the difference method. The degree of fit into the given node displacements was correlated with internal reliability of a network variant in a similar way as in the difference method. The discrepancies between the methods obtained in the tests suggest that, in general, a special attention should be paid to the choice of the evaluation method suitable for a given network.

The correlation between the robustness indices (shear and scale) and the level of internal reliability has been already reported in (Seemkooei 2001).

5. PRACTICAL USEFULNESS OF THE ROBUSTNESS INDICES IN EVALUATION OF THE NETWORK'S QUALITY

The strain-based robustness indices, termed the deformation measures, describe the behaviour of the differential surrounding of each individual network node. This surrounding, which does not physically exist, is a purely virtual extension of the network's structure. For the traditional reliability concept its link with network accuracy is easily established (Gruendig and Bahndorf 1985). As has been emphasized in (Vaniček et al. 2001) such a link between the robustness analysis and the covariance analysis does not exist, as they address different aspects of a network. Consequently, the deformation measures although may have intuitive interpretation, are difficult to be used for practical purposes (e.g. setting realistic robustness thresholds, improving the network structure to meet both the accuracy and the robustness requirements).

The weak sides of the strain analogy bring to mind the need for finding a non-tensor replacement for network robustness analysis, the network being treated as a discrete system. One of the possible approaches, applicable to all types of geodetic networks and sufficient for most of practical purposes, could be the use of traditional concepts of internal and external reliability. The robustness measures could be as follows:

- max. displacement of each network node due to maximum undetectable error in one of the observations, especially of those coming to this node; the displacement would be computed with the use of the specified datum constraints (*external reliability*);
- max. change in each observed network element due to maximum undetectable error in this observation (*internal reliability*).

  The change would be computed on the basis of the redundancy numbers and would be invariant to the changes of the non-distorting datum.

7. CONCLUDING REMARKS

It should be emphasized that the resorting to strain tensor analogy to analyse network robustness has been an interesting and innovative idea. However, the analogy is expected to yield the robustness measures of sufficient accuracy and robustness criteria interpretable in terms of the behaviour of geodetic networks themselves, but not of virtual systems being extrapolations developed for the sake of analogy. Looking from a purely theoretical point of view, we might add a weak side of the strain tensor analogy, that it does not seem to have chances to be developed into a generalized approach, i.e. applicable to all types of geodetic networks, with different types of datum constraints. Additionally, there can be the cases where for some points in a network the elements of the displacement gradient cannot be determined.

Undoubtedly, the strain analogy for the robustness analysis of geodetic networks re-quires further studies. Rigorous conditions for the applicability of this analogy to various types of networks (e.g. with different levels of internal reliability) should be worked out, with special attention being paid to methods of evaluating the elements of the displacement gradient. The findings in this area could be the basis for formulating the procedures for network design or improvement. On the other hand, a non-tensor replacement for the robustness analysis should be sought after.

Interesting suggestions as regards the applications in geodesy of the analogy based on mechanical models can be obtained by observing the evolution of methods for examining the

truss structures in civil engineering. Initially, the basic task was the analysis that used a discrete description of such structures (structure analysis), and was quite satisfactory for modelling of their behaviour. To apply this analysis it was necessary to form a mathematical model of the structure, taking into account its geometry, the internal and external acting factors, and above all, the properties of the elements connecting the structure's nodes.
In the course of time, there emerged a need to design a structure being optimal with respect to its weight (i.e. least-weight design). There came the methods of structure synthesis, aimed at examining all the trusses, the geometry of which as well as the properties of rods and the acting loads, fulfil prescribed conditions. A natural basis for such a class of rod structures was a continuum, satisfying given constraints. From the continuum, a discrete structure was obtained by optimal removing of the useless material (Holnicki-Szulc et al., 1995). To control the optimization process the concept of strain tensor was necessary.
The approach as above, seems to be worth considering in elaboration of the strain- analogy based procedures for the design of geodetic networks, optimal with respect to the number and distribution of observed elements as well as their accuracies.

References

Dare P. (1983) *Strength analysis of horizontal networks using strain*, Survey Science, Technical Report 2, University of Toronto, Erindale Campus, Mississauga, Ontario

Eringen A.C. (1967) *Mechanics of Continua*, J. Wiley, New York

Gambin W. (1977) *A method of large finite elements*, Comp. Math. Appl. Meth. Eng. 11, 351- 365

Gambin W. (1993) *Continuum approach to textured materials,* Proc.7th Int. Symp. on Continuum Models of Discrete Systems, Padeborn/Germany, 287-296

Gruendig L.,Bahndorf J. (1985) *Sequential optimisation of geodetic networks with respect to accuracy and reliability*, Proceedings of 7th International Symposium on Geodetic Computations, Kraków, Poland

Holnicki-Szulc J., Gierliński J.T. (1995) *Structural Analysis, Design and Control by the Virtual Distortion Method*, J. Wiley, Chichester.

Michel V., Person T. (2003) *From geodetic monitoring to deformation tensors and their reliability*, Proceedings of 11th FIG Symposium on Deformation Measurements, Santorini, Greece, pp. 463 - 469

Parzyński Z (2007) *Analysis of properties of the strain-analogy based robustness measures for geodetic networks* (in Polish), doctoral thesis (not published)

Prószyński W (1997) *Measuring the robustness potential of the least-squares estimation: geodetic illustration*, Journal of Geodesy 71: 652 – 659

Prószyński W (1994) *Criteria for internal reliability of linear least squares models*, Bull. Géodésique 68: 162 - 167

Seemkooei A.A. (2001) *Comparison of reliability and geometrical strength criteria*, Journal of Geodesy 75: 227 - 233

Vaniček P, Craymer MR, Krakiwsky EJ (2001) *Robustness analysis of geodetic horizontal networks*, Journal of Geodesy 75: 199 – 209

Corresponding author contacts

Witold PRÓSZYNSKI; wpr@gik.pw.edu.pl
Dept. of Engineering Surveying, Faculty of Geodesy and Cartography; Poland