Empirical influence functions of different robust estimation methods applied in displacement analysis

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ABSTRACT

Robust methods of estimation might have different foundations, necessary assumptions and hence also different general properties. Some robust estimators might also be applied in deformation analysis. Here, we consider three kinds of the robust estimates: basic M-estimators (the Huber or Tukey methods), M_p estimator (M-estimator based on the Pearson distribution family) and finally R-estimator (weighted Hodges-Lehmann estimator). Note, that the first two estimators belong to the same family of M-estimation; however, their theoretical assumptions differ much from each other. The last estimator considered here has completely different foundations (it is based on the rank tests), and hence it has different properties. To describe the properties of the estimators one can apply different ways or measures. The influence function is one of the most important and popular way to describe such estimators and it is also very useful while designing new estimators. Such a function provides just general information; however, from the practical point of view it is also important to know how the estimates behave in the case of a particular observation structure. Hence, it is also very advisable to analyze the empirical influence functions (EIFs) which might describe the behavior of estimates from many points of view and for various disturbances within the observation set. The paper presents EIFs obtained for different variants of disturbing errors which might occur in deformation analysis. They show that it is difficult to flag the best, "most universal" estimation method, and the choice of the most convenient method is sometime just impossible.

I. INTRODUCTION

There is no doubt that the least squares method (LS) is the most popular estimation method which is applied in geodesy or surveying. This is due to its simplicity but also well-known theoretical properties, just for example, LS estimates are BLUE (best linear unbiased estimates) or MVUE (minimum variance unbiased estimates) if only measurement errors are normally distributed. On the other hand, LS estimates are not robust against outlying observations (e.g., Prószyński, 1997; Baselga, 2007; Duchnowski, 2011) and hence there is a need for other estimators which are robust. We usually assume that outlying observations are just observations affected by gross errors; however, outliers might have different origins (e.g., Duchnowski, 2011; Duchnowski and Wiśniewski, 2017a). Robustness is also important in deformation analysis where outliers might result, for example, from instability of some reference points (Duchnowski, 2011). There are several different classes of robust estimation, and hence several different approaches to robust estimation (e.g., Huber, 1981). The most popular robust methods, which are applied in geodetic or surveying problems, usually belong to the class of robust M-estimation, and the respective optimization problem itself is usually solved by applying IRLS algorithms (iteratively reweighted least squares). Here, one can list many methods, such as Huber's, Hampel's, Tukey's or Danish method. The other variant of M-estimation was proposed in (Wiśniewski, 2014). It is based on the particular probabilistic models of observation errors, namely Pearson's distributions. This allows us to consider both leptokurtic and platykurtic distributions in such a context. Note, that $\mathrm{M}_{\mathcal{P}}$ estimation as well as its respective estimates (here called MPEs) have shown some robustness against outliers (Wiśniewski, 2014; Duchnowski and Wiśniewski, 2017a).

The other approach to robust estimation is applied in the case of R-estimation. The estimators of such a class are generally based on rank tests or specially defined score functions. The basic R-estimates are called Hodges-Lehmann estimates (HLE), and Duchnowski (2013) proposed another variant which was called weighted Hodges-Lehmann estimator (HLWE), and which can also accept observations of different accuracy. The estimators in question can be applied in deformation analysis (see, Duchnowski, 2010; 2013) or in other geodetic problems (for example, Kargoll, 2005). It is also worth noting that HLWEs (or respective HLEs) might have higher accuracy than classical LSEs (see, Duchnowski and Wiśniewski, 2017b). This concerns especially observation sets which empirical excess kurtosis is positive (in other words, observation errors have leptokurtic distribution). Similar conclusion also concerns MPEs (Duchnowski and Wyszkowska, 2017).

We can apply many measures or ways to investigate a robust method. The basic and most important information about the method properties stem from its breakdown points and influence functions. It is worth emphasizing that such general information about the method is not enough sometimes. The specific features of the method can be investigated by applying empirical breakdown points or empirical influence functions (see, for example, Rousseeuw and Verboven, 2002; Duchnowski and Wiśniewski, 2017a). This allows us to understand how a method behaves in the case of a particular observation structure (a geodetic network, measurement epochs). This paper is focused on the empirical influence functions (EIFs) of the earlier mentioned estimators, namely robust M-estimates, MPEs and HLWEs. In the case of M-estimation we have chosen two methods: Huber's and Tukey's one, which differ from each other in the shape of the influence function. Most of all, Tukey's biweight function has symmetric finite rejection points. The main goal of the paper is to show an example computation for EIFs of different estimates for a chosen simulated levelling network. It might help us to decide which method will be most suitable for a particular problem or a particular disturbance of an observation set.

II. MODELS AND THEORETICAL ASSUMPTIONS

Let us assume the following classical functional model of geodetic observations

$$\mathbf{y} = \mathbf{AX} + \mathbf{v} \tag{1}$$

where: $\mathbf{y} = [y_1, \cdots, y_n]^T$ is an observation vector, $\mathbf{v} = [v_1, \cdots, v_n]^T$ is a vector of random measurement errors, $\mathbf{X}$ is a parameter vector, $\mathbf{A}$ is a known matrix of coefficients. Let all observations have the same accuracy, and their assumed standard deviation $\sigma = 1$ mm, hence the weight matrix of observations is equal to the identity matrix. Then the LS estimate of the parameter vector can be written as

$$\hat{\mathbf{X}}_{LS} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y} \tag{2}$$

The very similar equation can be written for all M-estimators, namely

$$\hat{\mathbf{X}}_{M} = \left(\mathbf{A}^T\mathbf{W}\left(\hat{\mathbf{v}}\right)\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{W}\left(\hat{\mathbf{v}}\right)\mathbf{y} \tag{3}$$

where: $\mathbf{W}\left(\hat{\mathbf{v}}\right)$ is a diagonal matrix which diagonal elements $\left[\mathbf{W}\left(\hat{\mathbf{v}}\right)\right]_{ii} = w(\hat{v}_i)$, and $w(\hat{v}_i)$ is a weight function related to the particular method, $\hat{v}_i$ is a standardized error of the *i*th observation. Note, that solution presented here is an iterative process which ends when the parameter vector is not changing between the iteration steps any more (or the change is smaller than the assumed tolerance). Considering the assumption presented here, the weight function of the Huber method can be written as (for example, Gui and Zhang, 1998; Baselga, 2007)

$$w_H(\hat{v}_i) = \begin{cases} 1 & for \quad |\hat{v}_i| \le a \\ \dfrac{a}{|\hat{v}_i|} & for \quad |\hat{v}_i| > a \end{cases} \tag{4}$$

a is a positive constant (usually assumed between 1.5 and 3.5). In the case of the Tukey method we have

$$w_T(\hat{v}_i) = \begin{cases} \left(1 - \dfrac{|\hat{v}_i|^2}{a^2}\right)^2 & for \quad |\hat{v}_i| \le a \\ 0 & for \quad |\hat{v}_i| > a \end{cases} \tag{5}$$

where this time the constant a is usually equal to or lower than 6. In the case of MP estimation, the weight function is related to the respective probabilistic model. In such a context, the Pearson distribution system seems very useful since it allows us to apply the known (or estimated) kurtosis and/or asymmetry of observation errors. Here, we assume that an error distribution is symmetric and leptokurtic. In such a case, the weight function can be based on Pearson's distribution of the type VII (Wiśniewski, 2014). MP estimation requires information about kurtosis of the measurement errors. One can apply known value of that parameter or estimate it from the observation sets (using, for example, empirical moments). However, if we assume an inflated value of the kurtosis then the method might accept also outlying observations, hence the estimation results would not be affected by such observations (Wiśniewski, 2014; Duchnowski and Wiśniewski, 2017a). Considering application of all methods presented here in deformation analysis, we can just estimate point coordinates at all measurement epochs separately. By subtracting the respective coordinate estimates we can assess displacement of all object points. Note, that such an approach is the basic one, and other methods can be found in (e.g., Hekimoglu *et al.* 2010).

The last robust estimation method which is considered here is R-estimation. In deformation analysis, we can apply the basic R-estimator of the shift which is a natural estimate of coordinate changes between measurement epochs (Duchnowski, 2010; 2013). Since geodetic observations might have different accuracy, application of the weighted Hodges-Lehmann estimates (HLWE), which were proposed by

Duchnowski (2013), are especially advisable. The general form of that estimator can be written as follows

$$\hat{\Delta}^{HLW} = medw\left(y_i - x_j\right) \tag{6}$$

where: *medw* is a weighted median operator (the ways of the weighted median calculation can be found in (e.g., Gurwitz, 1990; Duchnowski, 2013)). Considering application in deformation analysis, y_i and x_j are coordinates of a particular point which are computed independently at the second or first measurement epoch, respectively (the basis for such coordinates are measurements at respective epoch and coordinates of the reference points). Note, that for a particular point, there are always several ways to compute its coordinates independently. The number of such ways is always limited; hence we never use all measurements for computing each point coordinates. We can also apply the variant of HLWE proposed in (Wyszkowska and Duchnowski, 2018). In such a variant more measurements are applied for each point, and hence the estimate is more reliable and accurate. For example, in the case of levelling network some height differences can be computed in several independent ways. In the variant of HLWE in question, one can compute the necessary height differences as an average of such ways. Note, that the final heights of the particular point, which are applied in Eq. (6), should still be independent.

Let us now consider the most appropriate form of an empirical influence function in the context of the paper goal. Such a function can be defined in several different variants depending on the assumed functional model or the way of gross errors occurrence (e.g., Rousseeuw and Verboven, 2002; Duchnowski, 2011; Duchnowski and Wyszkowska, 2018). Here, the following form seems the most appropriate

$$\mathrm{EIF}(x) = T_n\left(\mathbf{y}_1 + \mathbf{g}_1, \mathbf{y}_2 + \mathbf{g}_2\right) \tag{7}$$

where: T_n is a tested estimator, $\mathbf{y}_i$ are observation vectors, $\mathbf{g}_i$ are vectors including gross errors at the respective epoch. Note, that the variable x appears only in $\mathbf{g}_2$. Thus, we propose the following structure of the gross error vectors

$$\begin{aligned} \mathbf{g}_1 &= \begin{bmatrix} g_{11} & \cdots & g_{1l} & \cdots & g_{1n} \end{bmatrix}^T \\ \mathbf{g}_2 &= \begin{bmatrix} g_{21} & \cdots & x_k & \cdots & g_{2n} \end{bmatrix}^T \end{aligned} \tag{8}$$

where: g_{il} stays constant for each computed EIF, and $x = x_k$ varies from -25 mm to 25 mm and it affects only the kth observation at the second epoch. Note, that in the case of the conventional EIF all values g_{il} are equal to zero. However, if one wants to investigate how the estimator behaves in the occurrence of multiple gross errors then some g_{il} should be non-zero.

III. EMPIRICAL TESTS

Let us consider a leveling network presented in Fig. 1 which is established to test vertical displacements of the object points 1, 2 and 3. Let the network be measured at two different measurement epochs and let standard deviation of all measurements σ_{h_i} = 1.0 mm. Let us now simulate the vectors of random errors for the both epochs as follows (all in mm)

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} -0.5 & 0.8 & 2.3 & 3.1 & -1.4 & -0.8 & -0.7 \end{bmatrix}^T \\ \mathbf{v}_2 &= \begin{bmatrix} 1.0 & -1.8 & 0.0 & 0.1 & 1.4 & -1.0 & 0.2 \end{bmatrix}^T \end{aligned} \tag{9}$$

We assume that all reference points are stable, and the object points are displaced between the measurement epoch. Here, we take $\Delta H_1 = 10$ mm; $\Delta H_2 = -20$ mm; $\Delta H_3 = 5$ mm.

Such assumptions allow us to compute EIFs for several different variants of the vectors $\mathbf{g}_i$. They are also useful for assessing the most suitable values of the steering parameters, namely *a* for the Huber or Tukey methods, and β (simulated kurtosis) for MP method. The respective EIF (obtained for $x = x_1$) for the first two methods are presented in Fig. 2 and 3. The figures show that for different values of *a* we can obtain little bit different shapes of EIFs. Since we want the methods to be robust against outliers, we should take such *a* for which EIF is bounded. On the other hand, the EIFs should be quite "smooth" not to be influenced much by the particular simulated vectors of the random errors, Eq. (9). For the Huber method for *a* = 2.5 EIFs are unbounded (in fact they are the same as for LS method). Thus, for that method we take *a* = 2. As for the Tukey method for *a* < 3 EIFs are not "smooth" for relatively small absolute values. Thus, the choice of *a* = 3 seems the most appropriate (EIFs are "smooth" and seem bounded at least for the considered values of *x*).

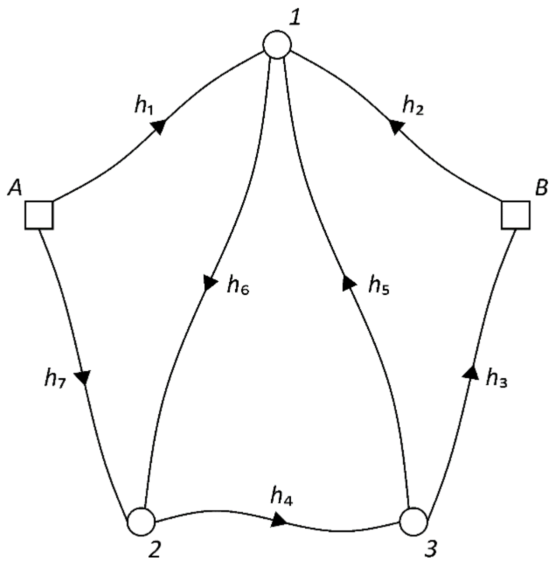

Figure 1. Tested levelling network

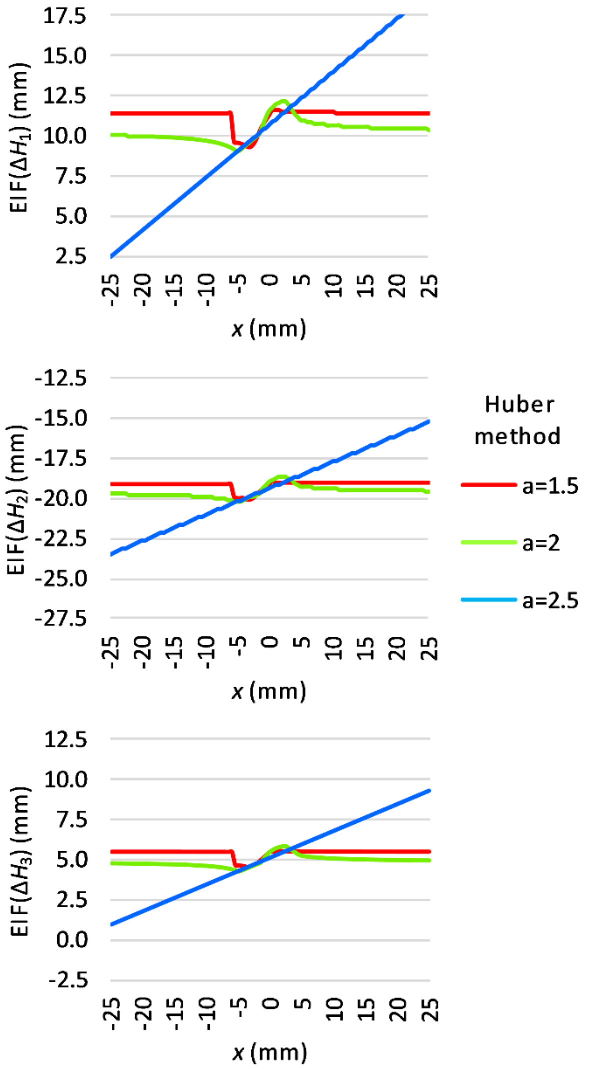

Figure 2. EIFs of the Huber estimates for different *a*

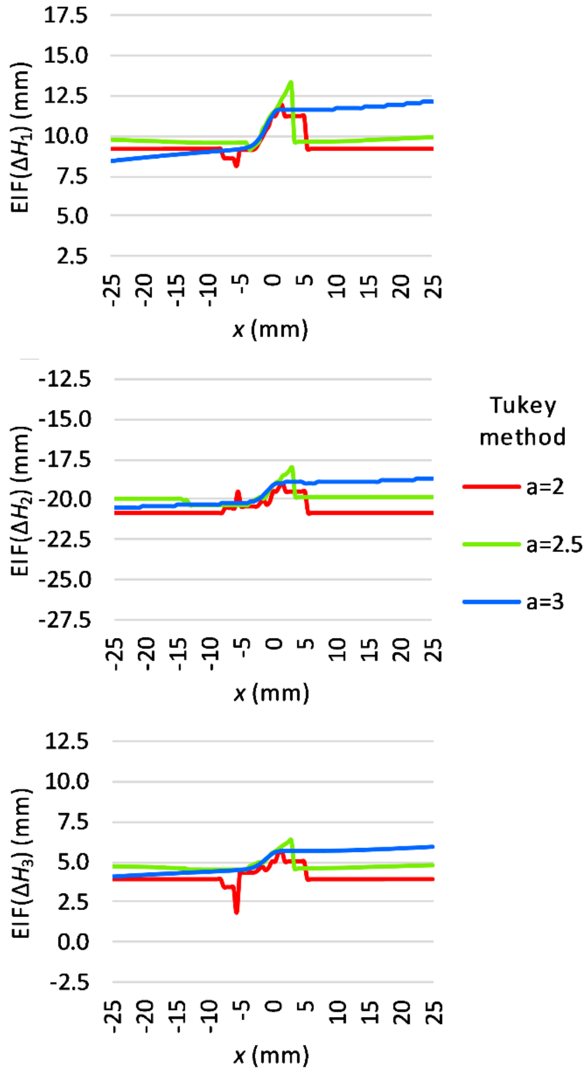

Figure 3. EIFs of the Tukey estimates for different *a*

The example presented here shows that EIFs can be also useful when assuming the value of the constant *a* which is the most appropriate for a particular network or, more generally, a geodetic observation set. The other ways of computing or assuming such a value can be found in (e.g., Gui and Zhang, 1998; Berber and Hekimoglu, 2003). As for MP method, all EIFs which are obtained for different $\beta \in \langle 3, 15 \rangle$ are very similar for one another, hence we do not present them here. We finally decide to assume that $\beta = 15$ which allows the method to accept outlying observations with the bigger random errors (see, Wiśniewski, 2014). Note, that such a large kurtosis is just unusually in practice. In the case geodetic or surveying observation sets β is usually lower than 4 and very seldom higher than 8 (see, e.g., Wiśniewski, 2014). Here, it does not reflect the nature of the observation set but is chosen to make MP estimation robust against outliers.

Now, let us assume several different variants of the gross error vectors and discuss how the estimators under investigation behave in the occurrence of a single or multiple gross errors. The variants differ from each other in the location of the varying x_k (different k) and location of constant gross errors g_{il}. Note, that in the following description of the variants we indicate only non-zero g_{il}.

A. *Variant: k = 1*

Fig. 4 presents EIFs which are obtained for all estimators. It is clear, that respective EIFs obtained for different points and the same estimation methods are similar to one another; however, some discrepancies are also vivid. The most evident finding is that EIFs of MP method are not bounded, hence the method is generally not robust. Nonetheless, EIFs of MP estimates are similar to the other EIFs for small values of $x = x_1$ (from -5 to 5 mm). Moreover, EIFs of R-estimates are similar to EIFs of the Tukey method. It is also evident that the Huber estimates are most robust, their EIFs are the closest to the simulated values of the displacements for all object points.

B. *Variant: k = 1,* $g_{27} = 0.01$ *m*

EIFs of that variant are presented in Fig. 5. In the case of multiple gross errors, EIFs have more complicated shapes and their value spread is larger. This reflects the negative influence of outliers on the estimated values. Generally, if the absolute value of $x = x_1$ is the biggest then the results of all estimation methods are unsatisfactory (they are far from the theoretical values). In the case of the point 1 all EIFs are very similar to each other. For the point 2, EIF of HLWE has the biggest value range. On the other hand, that estimate is not influenced by the negative $x = x_1$ (such a property is natural for R-estimates (see, Duchnowski, 2011)). For the point 3, HLWE is the best and its EIF is almost constant for all considered here values of $x = x_1$.

C. *Variant: k = 1,* $g_{17} = 0.01$ *m,* $g_{27} = 0.01$ *m*

This time we simulate occurrence of outliers in both measurement epochs. The respective EIFs are presented in Fig. 6. The comparison between EIFs in Fig. 5 and Fig. 6 seems especially interesting. It shows that the respective EIFs of HLWE differ from each other most. What is more, when there is an additional outlier at the first measurement epoch, then HLWEs are generally less affected by all outliers (see, especially points 1 and 2). As for the rest of the estimates, EIFs are a little bit changed and usually the additional outlier at the first epoch worsens the estimation results. It is also worth noting than for many values of $x = x_1$ MP estimation is less affected by the outliers than both robust M-estimates.

D. *Variant: k = 4*

In that variant, the observation h_4 at the second epoch is outlying. Thus, we check how location of a single outlier affects EIFs of the estimates under investigation. The respective EIFs are presented in Fig. 7. It is advisable to compare such EIFs with EIFs presented in Fig. 4 and related to the first variant with only one outlier too. It is clear that only for the point 3 respective EIFs obtained for the same types of estimates are similar to each other. The shapes of EIFs for the points 1 and 2 are quite different at both variants, respectively (it concerns especially the point 1). The location of outlier is the reason, here. The observation h_4 is located relatively far from the point 1, hence it has much lower impact on the estimation of that point displacement. It is also worth noting that all estimation methods yield almost the same results; however, MP estimate is better than the rest. On the other hand, for

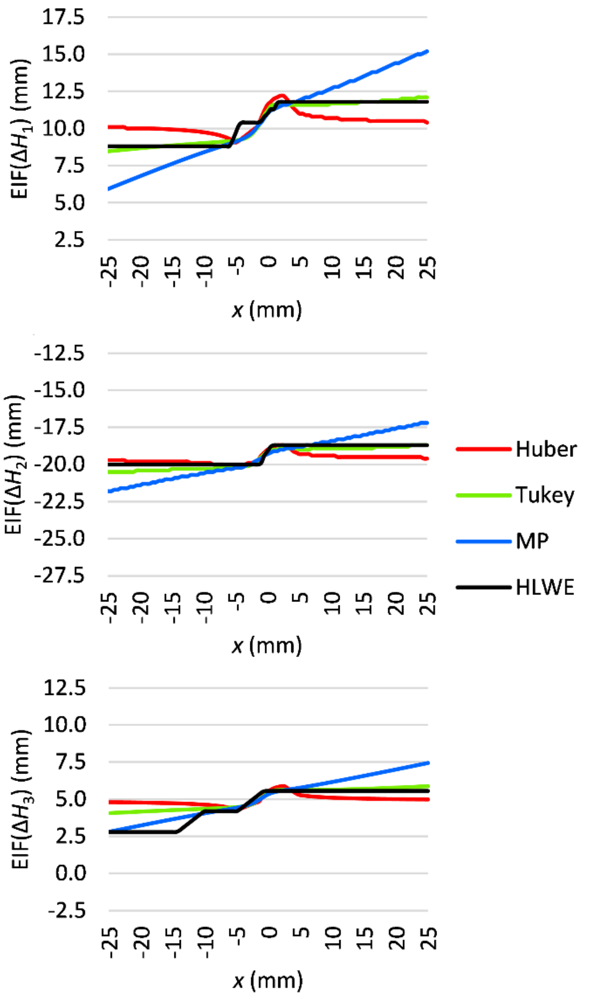

Figure 4. EIFs for variant A

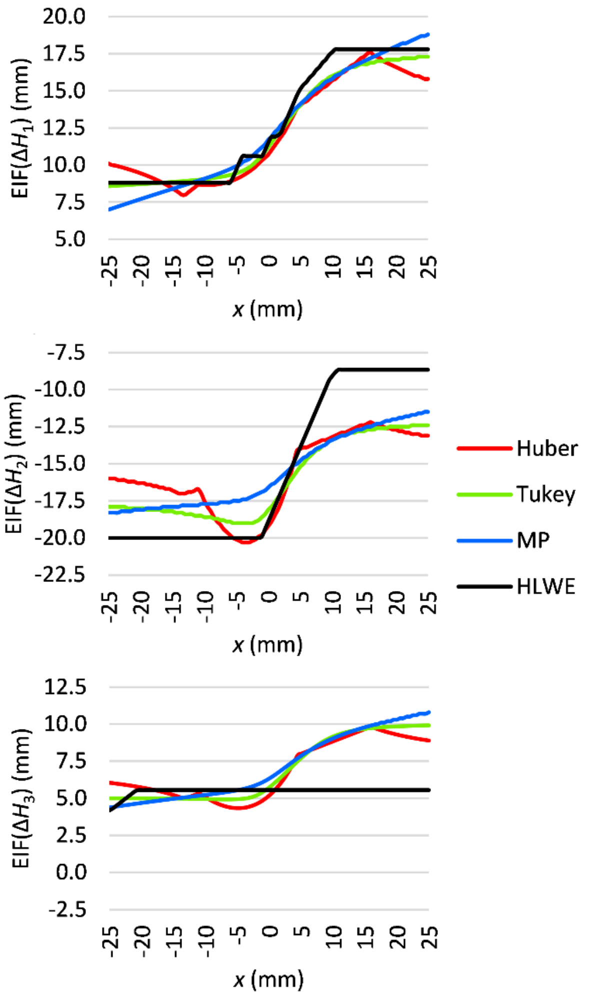

Figure 5. EIFs for variant B

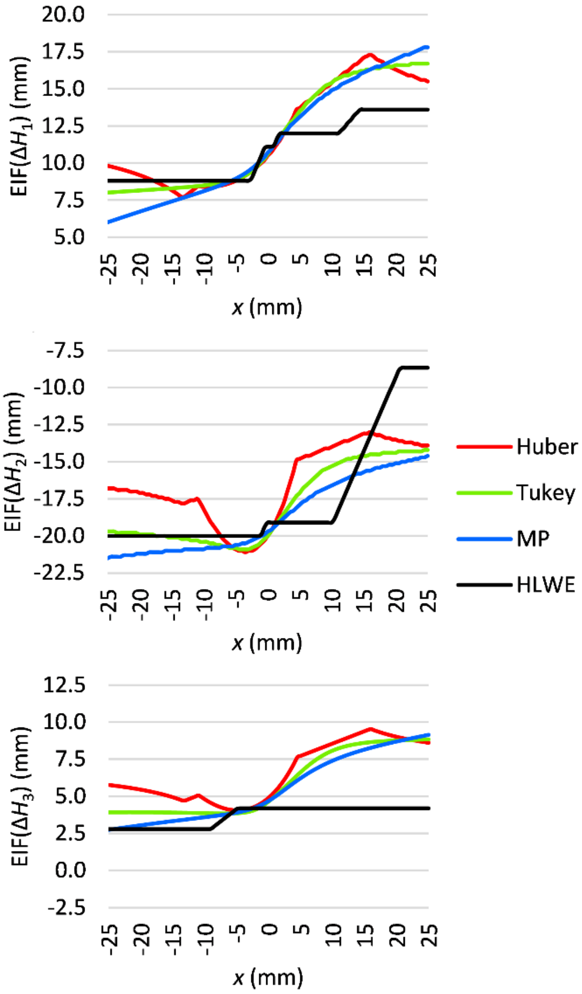

Figure 6. EIFs for variant C

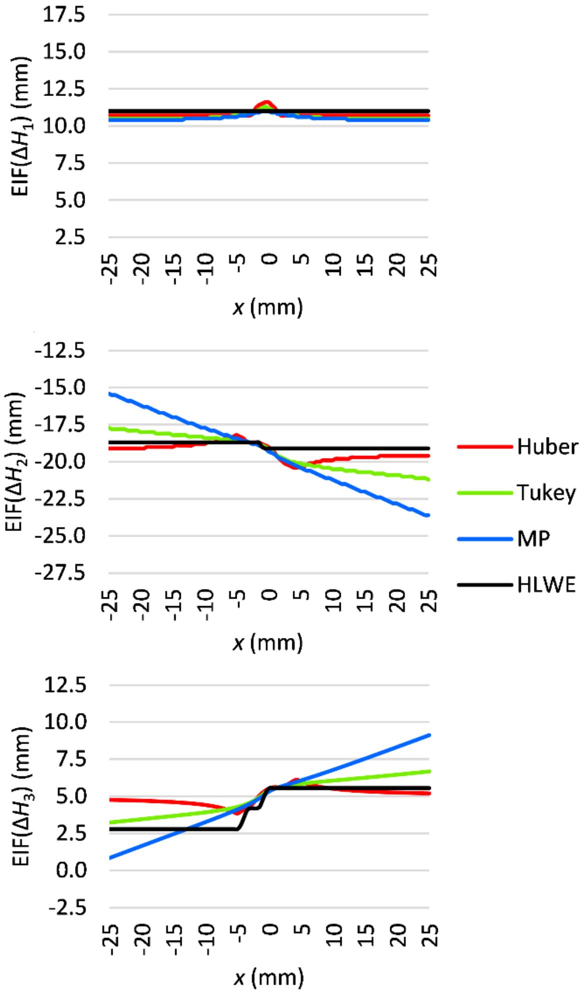

Figure 6. EIFs for variant D

the points 2 and 3, such an estimator shows almost no robustness and the respective estimates are strongly affected by the outlier.

IV. Discussion

The main goal of the paper was to present general properties of several robust methods by computing empirical influence functions for several different variants on the number and location of outliers. The tested network is relatively small, thus in fact all robust methods can withstand only one outlier (one can say that it is their empirical breakdown point). The respective EIFs show that almost always they break down when there are two outlying observations. HLWE seems to be an exception here; however, it stays robust only in the case of one object point. Such knowledge is also useful from the practical point of view. By computing and comparing the respective EIFs, one can predict which estimates of object point displacements are mostly exposed to the bad influence of outliers. In some sense it is similar to the analysis of network reliability (Prószyński 1997).

The comparison between EIFs for different estimates shows that it is hard to point out the best estimator (the estimate which is the most robust or which values are affected by outliers least). In the case of a single outlier (Variants A and D), the Huber estimates seem to be the best choice. For the multiple outliers all estimates at hand break down, and only HLWEs can provide satisfactory results. Note, that even if the results are affected by outliers, the estimated displacements are always relatively far from the simulated one. Thus, we can always conclude that all object points are not stable.

MP estimates seem to be the least robust or not robust at all, and hence they are the most sensitive to outliers. However, there are also some cases in which such estimates provide results that are very similar (or even better) to the results of the other methods. Note, that MP estimate can be more robust if one applies asymmetry coefficient during the estimation process (see, Wiśniewski, 2014; Duchnowski and Wiśniewski, 2017a).

V. Conclusions

The paper shows some comparisons between methods which present different approaches to robustness against outliers and their basic application in deformation analysis. Note, that the methods chosen here have not been compared with each other in such a context yet. The collation of findings points out that the Huber's or HLW estimates are the most advisable and useful from the practical point of view.

The empirical analyses show that the shapes of EIFs for the presented methods differ from each other in the case of a single outlier, and they are more similar for the multiple gross errors. Note, that EIFs for the same method might vary between different network points (different estimated parameters). Thus, EIFs might describe how the estimates behave in the case of particular observation set, network or estimation problem. That is the advantage of EIF over the theoretical influence function (IF) which provides only general information about the estimation method.

The paper presents also another application of EIFs. Such functions might be useful when one chooses values of the steering parameters for some robust methods. Note, that only HLWE does not need any assumption in that context. The assumption of an inappropriate values of parameters might lead to the loss of robustness (usually, a method cannot identify outliers). We can also use the steering parameters which results from the theoretical analysis of the method properties. However, by applying EIFs one can choose the best values of the steering parameters which are related to the particular network or other observation structure.

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