

On Comparison of the Three Meridian Arcs in Lapland

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SUMMARY

A research sketch is presented of actual accuracies of the three historic meridian arc measurements in Lapland (performed by Maupertuis, Svanberg and Selander in XVIII–XIX cc.) based on the original treatises by the workers. Previous investigations of the same matter, mathematics and up-to-date geodetic data are used. The astronomical and geodetic errors of each measurement are summarized.

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1. INTRODUCTION

The historical Lapland provides a rare example of three important geodetic works successively performed in the same country: those are the meridian arc measurements directed by P.L.M. de Maupertuis (1698–1759), J. Svanberg (1771–1851) and N.H. Selander (1804–1870). Historic significance of the first (particularly) and second works is well popularized, the third one is not generally known. Anyway, the technicals, results and state-of-the-art accuracies of those present a specific subject of the history of geodesy. This research aims at wider awareness of the actual achievements of the three historic measurements *among surveyors*. It will use the method of cross-comparisons and base on the original treatises and posterior re-measurements, especially, those accumulated within up-to-date geodetic databanks. Selander's work is of particular interest among the three for it was carried out in the frames of the great trans-European project by F.G.W. Struve (1793–1864), in 2005 acknowledged as the first multi-national World Heritage – “*the Struve Geodetic Arc*” (SGA).



Re-measurements is the exact method to judge on quality of historic arc measurements. For the purpose authentic points of the works should have been restored first. It is definitely known that all the eleven Maupertuis's stations have been marked on ground, many were engraved in bedrock [Maupertuis 1738, pp. 15, 29–30, 121]. For instance, Svanberg has identified Maupertuis's two crosses in bedrock at the site of *Niemisby* (the southern end of his baseline; further on *station names* are in the original spellings only); in his turn, he engraved a new cross adjacently and measured distances between all of them [Svanberg 1805, pp. 19, 22; figures 17, 26, 27]. He also marked the northern end of his baseline with a pole buried at the site *Poiki Torneå* 14 km apart. At present *none*

of those markings is aware of. As to Selander's ones, two original bedrock engravings survive at his northern terminal *Stuor-oivi*; at his other measurement sites markings, as it seems, have not been made at all [Leinberg 1929, p. 158], except for, may be, a bedrock cross found at *Perra-vaara*. At present *Stuor-oivi* and *Perra-vaara* are members of the World Heritage SGA. Another type of authentic points of the surveys in view are surviving church towers. Luckily, two of them remain in good condition and are widely known: they belong to the town and district churches in Tornio, Finland.

2. GEODETIC QUALITIES

2.1 Points of Reference

To be sure about identity of points of the three arcs involved, further on *triangulation station centres* will be referred to, as is customary in practice of surveying. Maupertuis observed at *Torneå* (now Tornio) 143.7 m south of the town church bell-tower spire (further on the latter is denoted T), and at *Kittis* 7.4 m north of the centre of the wooden signal (it is denoted Q). Svanberg observed latitudes at additional stations *Mallörn* (μ) and *Pahtavara* (π) situated far south and far north of T and Q correspondingly; however, he was able to *calculate* the distances between the parallels of T, T* and Q* (this latter was established on the *Kittis* hill supposedly close to Maupertuis's Q). Selander and Struve's collaborators observed latitude in the vicinity of the *District Torneå* (now Alatornio) church 2.24 km south of T, whereas one of Selander's points, a mere triangulation signal (Q**) was established on the same *Kittis* hill top supposedly close to Maupertuis's Q and Svanberg's Q*. This study will be leant on the surviving T and T* – two definite points common to all the three historic measurements; they are shown at the bottom of the map. Seven common sites north of T and T* form an interesting polygon for cross comparisons, although neither Svanberg nor Selander built their signals there with care to explanations about superposing signal centres to those of the previous measurements. In 1928 an on-spot investigation by Y. Leinberg has ascertained rather a close vicinity of probable positions of the northern terminal stations Q, Q* and Q** on the *Kittis* hill top, with a possible error of some metres in the latitudinal direction, within 10 metres anyway [Leinberg 1929, pp. 152–153].

2.2 Original Results

The following final results obtained by the mentioned geometers will be referred to:

the 1st Lapland measurement performed from T to Q (1736 till 1737): the measured meridian arc, or distance of the parallels of T and Q = “54942.57 toises”; then, the correction for convergence of the meridians of T and Q (“+ 3.38 toises”) added; thus the measured distance of parallels of T and Q = 54945.95 toises, or, quite enough, just 54946 toises [Maupertuis 1738, pp. 92–93];

the 2nd Lapland measurement (1799 till 1803) within its section between T and Q* (60% of the entire stretch from μ to π):* the measured distance of the parallels of T and Q* =

“107039.635 m = 54919.2537” toises, the measured distance of the parallels of T* and Q* = 109279.753 m = 56068.600 toises; the latter value was not presented by Svanberg as a specific value, here it is obtained in the same manner as Svanberg’s former one: by subtracting and applying the same transformation coefficient [Svanberg 1805, p. 170–171];

the 3rd Lapland measurement (1845 till 1852) stretching from T to Stuor–oivi (S, at the border with Norwegian Finmarken;), both as a whole and within its southernmost section from T* to Q** (34% of the entire stretch):*

- a) the measured length of the geodesic joining T* and S = “166173.119 toises”, after Selander; Struve has re-computed the result for “166173.82 ± 1.64 toises” (probable error);
 - b) the measured distance of the parallels of T* and S = “163221.904 ± 1.689 toises” after Struve;
 - c) the measured length and direction of the geodesic joining T* and T = “1194.1 toises, azimuth (T*T) = 344° 25′ 04″.2” after Struve;
 - d) the measured length and direction of the geodesic joining T* and Q**:
= “56172.998 toises, azimuth (T*Q**) = 356° 35′ 20″.266 ”,
- all these after [Struve 1860, §§ 85.4.a.T; 89.4; 89.6; 115.9; 119; 121].

A standard geodetic computation can be made with the data cited under **c** and **d**, the Struve latitude of T* = 65° 49′ 44″.57 (see the Part 3.2 below) and the reference ellipsoid derived and in use by Struve (semi-major axis = 3272539 toises, flattening ratio = 1/ 294.7 [Struve 1860, § 27]); it comes to the following additional values:

- e) the distance of the parallels of T and T* = 1150.15 toises;
- f) the distance of the parallels of T* and Q** = 56069.50 toises;
- g) the distance of the parallels of T and Q** = 54919.34 toises.

It is a matter of no doubt whether Struve could have calculated the values **e**, **f**, **g** by himself, if that had been found necessary.

2.3 Comparative Study

The 1736–1737 arc measurement was first estimated by Svanberg on the ground of his new measurements in Lapland; he, actually, disavowed the former one; in 50 years Selander’s measurements provided the third set of observational data in the same country; Leinberg estimated the Maupertuis geodetic errors on the ground of Selander–Struve’s data; finally, the present–date geodetic databank of the National Land Survey of Finland (NLSF) provides the most reliable source. Further on, for reasons of better hang of the argument the historic measurements will be considered in this succession: 3rd, 1st and 2nd.

2.3.1 Measurements of the 3rd Expedition

A standard computation with relevant data from the NLSF databank makes sure that the SGA terminal point *Stuor–oivi* (S) remains at its original position accompanied by two bedrock crosses, although the modern cairn may have been shifted within decimeters because it is a posterior construction. Preservation of the identical SGA terminal points allows to transform their present–day coordinates into the following values which can be regarded nearly “true”:

aa) 166178.3 toises;

bb) 163225.2 toises;

these match the same items as specified under **a** and **b**. The values **aa** and **bb** correspond to some quasi-geoidal surface representing what Struve called “*the sea level*” in Lapland; at present the surface can easily be reached with known values of geoid’s heights. Given the “*true*” values, the following can be stated:

the Selander–Struve geodesic joining T* and S has an aggregate error of the length; it can be estimated as $(166173.8 - 166178.3 =) - 4.5$ toises (9 metres), or ca $- 1/ 37000$; as to the length error of the meridian arc joining the parallels of T* and S, it is $(163221.9 - 163225.2 =) - 3.3$ toises (6 metres), or ca $- 1/ 49000$ (of course, the meridian quantity is less).

Actual triangle sides had, may be, greater negative errors; e.g., those of the northern Lapland triangle of *Stuor-oivi*, *Kersti-vaara* and *Ounas-tunturi* seem to be about $- 1/ 25000$, irrespective of uncertainty caused by a posterior change of Selander’s original constructions. Relevant details of that historic geodetic work can be noted as well. One short on-ground baseline (ca 3 km, or $1/ 109$ of the entire stretch; it is shown on the map in red) has been measured twice with the Struve bar apparatus arriving at practically the same resulting value (the difference was about 1 ppm). Directions rather than angles were measured with the Littman theodolite having a 12-inch circle with 4 micrometers whose readings were cited up to $0''.001$; however, the root mean square (rms) error of an arbitrary differential angle was found rather big ($\pm 1''.6$) due to, as Struve reasonably noted, lack of the additional direction-checking tube. That was the reason why Struve has “re-built” and re-computed the entire Selander triangulation, coming to a reduced ($\pm 1''.25$) rms error of one angle and a bit longer distances than those computed by Selander previously [Struve 1860, §§ 96–98]; e.g., see the item **a** above. Nevertheless, the finalized SGA Lapland triangles came to be 1.4 times less accurate than those in Finland. The main causes of that were the same as in the Norwegian SGA segment: lack of the second tube (that shortcoming *could not* be fully compensated with mathematics), problems of pointing at Selander’s targets (big barrels established on high poles) and big distances between the stations [Kaptüg 2006]; worsening of centre reduction values as well, even in Finland [Struve 1860, § 86] where the targets were peaked.

It would hardly be correct supposing a homogenous scale error of the entire Selander–Struve meridian arc stretch which can be found as the value (**bb** / **b**); this arc will be taken just as a reference sample of quality when bringing the other two historic results into comparison; the aggregate errors of those can be estimated by referencing to the result just obtained.

2.3.2 Measurements of the 1st Expedition

The aggregate length error of the Maupertuis meridian arc joining the parallels of T and Q can be estimated by the value (see the Part 2.2, item **g**): $(54946 - 54919 =) + 27$ toises, or ca $1/ 2000$; in terms of the metre: $+ 53 \pm 10$ metres; ± 10 m is considered as the maximal uncertainty due to unknown position of the reference point Q relative to that of Q** (explained above). Leinberg estimated the geodetic error of Maupertuis’s meridian arc by the

value + 23 toises [Leinberg 1929, p. 152] basing on the SGA length standard; his estimate lacks details of the procedure, especially regarding use of Maupertuis's "3.38 toises" (cited in the Part 2.2).

The entire range of Maupertuis's 12 different ways of assembling the length of the quested meridian projection of the line TQ [Maupertuis 1738, pp. 89–91, 109–110, diagrams] proved to be *54 toises wide*. Importantly, *the mean* of those different results has *not* been taken; instead, Maupertuis has chosen two similar values at the very upper end of the scatter, on a mere geometrical reason. Ironically, the errorless value was within the dispersion very near to its middle value. The wide scatter of final values surely originated in the measurement techniques. The angles were observed by two-foot quadrants readable up to 1" with a micrometric device; however, resulting errors by up to *two dozens of arcseconds* were committed, mostly positive and, may be, of a systematic nature [Leinberg 1929, p. 149]; importantly, Maupertuis *himself ascertained* that such errors would shift the resulting length value by *scores* rather than *single units* of toises [Maupertuis 1738, pp. 111–112]. The 14-km baseline (it is shown on the map in yellow and blue) was measured across a frozen river stretch with wooden poles (ca 10 m long) ending with projecting nails, the poles were put immediately on the ice surface. Two different teams have come to a very small resulting disaccord of ca 1/ 133000; however, neglect of steepness of the baseline caused a positive scale error by about 1/ 10000 [Svanberg 1805, pp. 20–22 and 171]. The scale error had a minor influence though, for angular and azimuth errors of the Maupertuis triangles exercised a much greater effect. Yet the significance of Maupertuis's length error was lower than that of his astronomical fault (see the Part 3.3).

2.3.3 Measurements of the 2nd Expedition

Having completed his *new* measurement in Lapland Svanberg found Maupertuis's value of the meridian arc joining the parallels of T and Q oversized by "26.7 toises" [Svanberg 1805, p. 171]. As just mentioned, within this value Svanberg gives accent to "+ 5.355 toises" accumulated as a scale error. Svanberg measured nearly the same baseline (shown on the map in yellow and blue), and measured the same meridian arc segment (TQ*) which he has collated with TQ of Maupertuis. The accuracy of Svanberg's collation (one place of decimals of the toise) testifies close vicinity of the positions of Q and Q* despite lack of his verbal witness to that. A good agreement of Svanberg and Struve's results both in evaluating the Maupertuis error and length values of their common arc segment (56068.6 and 56069.5 toises correspondingly, see the Part 2.2) testifies absence of express shortcomings of the geodetic part of Svanberg's work. The collation with the SGA Lapland segment suffers, however, the above-mentioned uncertainty (within ± 10 metres) due to the difference of the actual positions of the respective points Q* and Q** on the *Kittis* hill top.

Another two estimates of the accuracy achieved by Svanberg can be obtained with the use of remaining buildings. Struve compared the length values related to a short 18-km line joining the church towers (T*) and *Kemi* – two identical targets of Struve and Svanberg's measurements. His comparison showed a disaccord by 0.6 toises between the corresponding

values (that by Svanberg was found shorter [Struve 1860, § 85]); however, the Svanberg value was actually just a spin-off from his principal scheme; while observing from two adjacent points he fixed *Kemi* church with two sightings only [Svanberg 1805, p. 55–56, triangulation scheme], therefore, necessarily, with a lower accuracy. Then, a comparison of Svanberg and Struve’s values of the distance of the parallels of T and T*: 1149.35 (extracted from Svanberg’s final table [ibid., p. 170]) and 1150.15 (see the Part 2.2, item e) comes, again, to the same small difference under one toise (ca 2 metres). Finalizing the analysis of the Svanberg geodetic work, the following can be stated:

there is no single evidence of express shortcomings of the geodetic part of the Svanberg work; the accuracy of his geodetic measurement of the meridian arc segment joining the parallels of T and Q* is near to that achieved afterwards by Selander with better instruments.

Some details would be to the point. Svanberg’s wooden targets were ingenuities by construction and surely surpassed in fineness those by Selander. While re-measuring the Maupertuis baseline across the same frozen river stretch, Svanberg used iron bars (of 6 toises, or 11.7 m long) and new French length standards; he has thoroughly introduced corrections for temperature, steepness, centring and height. In the triangulation a new Borda-type repetition circle was used only, it was built in 1801 in Paris under supervision by Delambre, and fit for both horizontal and vertical sighting.

In 1818–1828 two similar instruments were used by Gen. Tenner’s staff while measuring the SGA segment in the Russian western governments; the instruments had divided circles of 13 to 14 inches supplied with four verniers readable up to 10" (up to 2".5 by ocular estimate). The observations were made *from high wooden towers* with resulting rms errors of one angle (result of 40 repetitions) from $\pm 0".9$ to $1".0$ [Struve 1860, §§ 9, 39].

Svanberg’s measurements were *ground-based* and, sometimes, they were made at only two stations of a triangle, that is, one angle was often computed instead of observed; the number of repetitions varied from 5 to 50. Within the historic Lapland heptagon only four triangles have been measured in full; the corresponding triangular errors are twice under $+ 1"$, $- 6"$ and $- 9"$ [Svanberg 1805, pp. 44 to 99] hence rms errors of angles *could* reach $5"$, those of single directions, accordingly, $3"$.

3. ASTRONOMIC QUALITIES

3.1 Control Values

In 1928 Y. Leinberg re-observed either of the Maupertuis astronomical points T and Q and determined their latitude values [Leinberg 1929, p. 157]:

at *Torneå*: “ $65^\circ 50' 56".21$ ” + reduction to T (+ 6.5 m, or + $0".21$), thus

$$\varphi (T) = 65^\circ 50' 56".4,$$

at *Kittis*: “ $66^\circ 48' 28".96$ ” + reduction to Q (+ 9.9 m – 7.4 m, or + $0".08$), thus

$$\varphi (Q) = 66^\circ 48' 29".0 \quad (“- 7.4 m” bases upon [Maupertuis 1738, pp. 92–93]).$$

The values are separated by $\Delta\varphi$ (QT) = $0^\circ 57' 32''.6$, which is the meridian arc amplitude joining the parallels of T and Q; the number of decimals corresponds to the obtained accuracy. The three values, nearly “true”, will be *control* ones in the further comparative analysis.

3.2 Observations of the 3rd Expedition

Latitudes and azimuths have been observed at the two arc terminal points T* and S by different teams: at *Torneå* by Pulkowa (Russia) astronomers Lindhagen and Wagner in 1851, and at *Stuor-oivi* by the Swedish professors Selander and Agardh in 1850–1851. While the former determination has been checked by Struve in every detail after original field registers, the latter has not: Selander submitted to Struve already computed values. The Lindhagen–Wagner result:

$$\varphi (T^*) = "65^\circ 49' 44''.57 \pm 0''.07 \text{ prob. error}"$$

[Struve 1860, p. LXVI] can be transferred, accurately enough, to the point T as soon as the latitudinal interspace between the two churches is known (see the Part 2.2, item e: 1150.2 toises, or $72''.4$), then the following value is achieved:

$$\varphi (T) = 65^\circ 50' 57''.0 ;$$

it exceeds that of 1928 by $0''.6$ (cf. in [Bonsdorff 1929]: “ $0''.8$ ”). Similar differences within one arcsecond have been observed at three other SGA stations in Finland, Norway and Russia [Bonsdorff 1929, Klingenberg 1931, Kaptüg et al. 2003]. Such small disaccords seem natural considering change of observation techniques and reduction methods over the time. At all but one SGA northern stations (i.e. in Finland and Norway) latitudes were observed after Struve’s instructions by Pulkowa astronomers; internal probable errors of those were estimated by $\pm 0''.06$ to $0''.08$ [Struve 1860, pp. Lxi to Lxx], therefore the maximal possible error might be $\pm 0''.3$. At *Stuor-oivi* the latitude values observed by Selander and Agardh have crossed each other by $1''$ [ibid., § 89.5], resulting from use of two *different* theodolites; this fact proves that shifts – not *errors* – due to use of *different* instruments could really extend up to $\pm 0''.5$ or even a bit more. There is no reason to doubt about absence of greater errors of the latitudes observed at T* and S. A mere subtraction comes to the following empiricism:

the amplitude of the 3rd meridian arc measured in Lapland between the parallels of T* and S can be regarded accurate within $1''$ as the highest.

Now to possible errors of azimuths (astronomical orientation). Struve stated that the probable errors of the astronomical directions observed at T* and S were within $\pm 0''.7$ [ibid., § 112] hence maximal error of those might be up to $\pm 2''$. In 1928 the *Lindhagen* astronomical azimuth at the SGA terminal *Fuglenaes* (of 1850) was re-observed; it was found the same within $2''$ [Klingenberg 1931]; much of the $2''$ can be ascribed both to changes in astronomical technique and re-construction of the original wooden target situated less than 9 km apart.

3.3 Observations of the 1st Expedition

Naturally, such a degree of precision could not have been achieved in 1736–1737. It is worth mentioning that Maupertuis was able to observe latitude values of T differing by only 0".8 and 1".8 (!) from that of 1928; for the purpose 2–feet and 3–feet *quadrants* have been used respectively. However, those very good results originated in wide scatters of single values reaching 8" and 11" correspondingly [Maupertuis 1738, pp. 131–139]; that is, proximity of both final results was but a luck. The required arc amplitude, naturally, was observed with a more developed instrument and, unfortunately, with a much bigger error proved with the 1928 astronomical re–observations. The original result [ibid., pp. 92–93, 125]:

$\Delta\varphi$ (QT, 1737) = “ 0° 57' 28".67 ” + reduction (– 143.7 m – 7.4 m, or – 4".88), thus

$\Delta\varphi$ (QT, 1737) = 0° 57' 23".8

was found too small by 8".8 compared with the control one [Leinberg 1929, pp. 150, 157]. Maupertuis’s astronomical instrument was a vertical 9 French feet (2,9 m) long telescope with a divided sector measuring meridian zenith distances of close–to–zenith stars with accuracies up to 0".1; the same star observed at two different stations was supposed to indicate their latitudinal amplitude. The major cause of the found error was uncontrolled instability of the position of the sector zero–point, especially in separate observation series made at the arc terminals; indeed, significant zero changes have been found with the help of analysis of the Maupertuis numerics [ibid., p. 160]. More importantly, positions of the zero *could not* have been controlled during observations due to the mechanical design of the instrument; upgrades of the construction of a similar instrument used by the *successful Peru expedition* have substantially prolonged its work [Krasovskiy 1942].

Maupertuis’s actual astronomical error should be counted somewhere *about* – 8".8; first, even if Maupertuis had observed an errorless value, substantial change of astronomic technologies over the 200 years *had to* result in a different value; secondly, as mentioned above, identity of the position of Maupertuis’s point Q was proved with a possible error of some metres in the latitudinal direction – this uncertainty influences one decimal place of the results in view. Thus the value – 9" (it corresponds to ca 140 toises in length) can be adopted as an approximate aggregate estimate of the error of the latitudinal amplitude determined by the astronomers of the French mission to Lapland. That was the major (and long suspected since) fault of the expedition.

As to errors of azimuth, Leinberg found Maupertuis’s result at *Torneå* disaccording by 1'.3 relative to the contemporary value provided with the Finnish state triangulation; he added that this error was of “*no noticeable influence*” on the final result [Leinberg 1929, p. 151]; Maupertuis regarded as “*too little*” the disaccord by 0'.6 he has come to [Maupertuis 1738, pp. 107–108]. However, in the course of successive calculations orientation of triangle sides other than the initial one had to accumulate errors of intermediate angles and thus get worse. While the specific case of the *Torneå* azimuth error had sure little influence on the meridian projection of that same side (*Torneå–Kakama*, azimuth 4° 19'), that was not so in other parts

of the survey. Inaccuracies of orientation by arcminutes (one or a few) had to result in errors by *several metres* in meridian projections of some of the Lapland heptagon lines.

3.4 Observations of the 2nd Expedition

In Svanberg's astronomical observations the method of measuring heights of the Polar star was used; his instrument was the above-mentioned Borda repetition circle; possible errors of observed directions, no matter horizontal or vertical, could reach *a few arcseconds* (see the Part 2.3.3). On-going improvements of the reference system of star declinations had the same degree of influence; that was testified by Gen. Tenner for the latitude observed in 1818 with a similar repetition circle [Tenner 1843]. Unfortunately, there is yet no evidence that Svanberg's terminal points (μ) and (π) survive, therefore a direct collation with later observations cannot be made. Just to have an estimate of Svanberg's accuracy two astronomical values can be collated: the above-mentioned latitude value observed by Lindhagen at T* ($65^\circ 49' 44''.57$) and the matching value *derived* by Svanberg ($65^\circ 49' 41''.73$ [Svanberg 1805, pp. 170–171]); the difference between them is $2''.8$.

To understand it properly, one should go into details. The well-known interrelation should be adduced to underline the difference between two kinds of latitudes:

$$\varphi = B + \xi \quad (1)$$

where astronomical latitude φ is observed on the earth physical surface, whereas geodetic latitude B is a computed quantity related to the surface of a reference ellipsoid, with an initial point given; the second member in the right part denotes a geomorphologic influence on the plumb line projected on the meridian plane; the influence exhibits itself as a shift (deviation) of the latitude relative to the ellipsoidal value B . Svanberg's value is based on his observation at the southern terminal ($65^\circ 31' 30''.26$ [ibid., p. 149], and some astronomical error (ε) has sure been committed here. Once the relation (1) is applied to Lindhagen's result at T* and Svanberg's one at (μ), their difference comes to:

$$\varphi(T^*) - \varphi(\mu) = \Delta B(T^*\mu) + \xi(T^*) - \xi(\mu) - \varepsilon \quad (2)$$

One *can admit* a good enough accuracy of the meridian distance measured by Svanberg from (μ) to T* via triangulation elements, that is, absence of significant geodetic errors – on the same ground as demonstrated in the Part 2.3.3. Then the geometric quantity $\Delta B(T^*\mu)$ can be transformed from its *measured linear* value = 33798 metres at the sea level [ibid., p. 170] into angular amplitude ($1091''.2$; Svanberg deduced $1091''.47$ on a smaller ellipsoid). Finally, applying Svanberg's value:

$$2''.8 = \xi(T^*) - \xi(\mu) - \varepsilon; \quad \text{otherwise: } \varepsilon = \xi(T^*) - \xi(\mu) - 2''.8 \quad (3)$$

This kind of estimation would discover the astronomical (observational) error of Svanberg's latitude of *Mallörn* as soon as *geomorphologic likeness* or *difference* between the areas of T* and (μ) is expressed in terms of latitudinal deviation of the plumb line.

A careful inference from the analysis is this: errors by few arcseconds in either of Svanberg's latitudes can be admitted.

A total loss of Svanberg's azimuth targets prevents from forming an independent estimate of this kind of his errors affecting orientation of the triangulation sides; he himself admitted a possible azimuth error of the geodesic (μ) – (π) by *one arcminute* and was sure it would not affect the final result for the terminals have been established on practically the same meridian

[ibid., pp. 24–25]. This Svanberg’s general evaluation is another confirmation of essentially *higher* accuracy of his *angles* (within *arcseconds*) which was found in the Part 2.3.3 above.

4. MATHEMATICS

A mathematical approach will be used to appreciate principal factors influencing qualities of the meridian arc measurements. The following is the basic part of the geometric interrelation between length and angular amplitude of a meridian arc segment traced on the surface of an arbitrary reference–ellipsoid:

$$S / \Delta B^\circ = (a / \rho^\circ) (1 - e^2) \{ (1 - e^2 \sin^2 B_m) \uparrow - 3 / 2 \} \quad (4)$$

Denoted are: $S, \Delta B$ – length and angular amplitude of an arc segment (the latter in degrees); a, e^2 – semi–major axis and squared eccentricity of a chosen ellipsoid, B_m – mean latitude of the arc segment, $\rho = 57.2957795$ (one radian, in degrees), the arrow denotes involution. At the latitudes of northern Scandinavia this equation is accurate within 2 ppm (half a metre for arcs not exceeding 3 degrees).

Applying the formula (4) to *actual measurements on earth’s physical surface* introduces some uncertainty due to, first of all, the difference of observed and ellipsoidal latitudes just considered in the Part 3.4; consequently, the left part of the equation (4) would present “not refined” values burdened with the following *hidden* quantities:

- overlooked errors of measurement and computation of the linear length of the arc;
- overlooked errors of astronomical observation and computation of angular amplitude of the arc, or difference of those committed during the observation of either of the terminal latitudes;
- difference of unknown quantities presenting distortions of either of the terminal latitudes due to local geomorphologic influence within some areas round either terminal.

If all these hidden quantities could be removed, the equation (4) would be exactly feasible. However, in the frames of “pure” astrogeodetic works the third of the mentioned quantities presents an *irremovable* component of the results and, as such, the principal cause of failure of many historical arc measurements. That is why a general estimate could only be achieved with using (4) in the supposed way. For that purpose one should extract the *measured* values from the original treatises [Maupertuis 1738, p. 125; Svanberg 1805, pp. 191–192; Struve 1860, p. Lxvi and § 121], those are:

Maupertuis: $\Delta\varphi = 0^\circ 57' 28''.67$, $\varphi_m = 66^\circ 19' 35''$, $S = 55\,023$ toises,

Svanberg: $\Delta\varphi = 1^\circ 37' 19''.57$, $\varphi_m = 66^\circ 20' 10''$, $S = 92\,778$ toises,

Selander–Struve: $\Delta\varphi = 2^\circ 51' 13''.83$, $\varphi_m = 67^\circ 15' 21''$, $S = 163\,222$ toises.

Now, the formula (4) taken on a reference ellipsoid, e.g., that of A. Bousdorff (1888) would come to the following values S^* (matching the observed astronomical result) or, alternatively, values $\Delta\varphi^*$ (matching the observed geodetic result):

Maupertuis: $S^* = 54\,811$ toises, or $\Delta\varphi^* = 0^\circ 57' 42''$,

Svanberg: $S^* = 92\,810$ toises, or $\Delta\varphi^* = 1^\circ 37' 17''.6$,
 Selander–Struve: $S^* = 163\,304$ toises, or $\Delta\varphi^* = 2^\circ 51' 08''.7$.

Comparison of the observed and computed values comes to the following *mathematical estimates*:

- Maupertuis’s arc length is, apparently, too big by up to 212 toises (1/ 260), or the amplitude looks “shortened” by up to approx. 13”;
- Svanberg “shortened” his arc length by up to 32 toises (1/ 2900), or the amplitude looks a bit too big, maximally by 2”.
- Selander–Struve’s arc length looks “shortened” by up to 82 toises (1/ 2000), or the observed amplitude is oversized by up to 5”.1.

The two alternatives – geodetic *or* astronomical error – are but extreme (speculative) cases; a truer picture has been found in the preceding parts of this study on the sound base of *instrumental re-measurements*. It is of importance that a change of the reference surface would involve insignificant change (within 2 or so arcseconds) of the “astronomical error” as deduced with the formula (4). That is why use of the equation (4) can help to detect important *details*; just two examples of that are the following. Whatever reference ellipsoid is chosen for (4), Selander’s arc exhibits a marked *excess* of the observed amplitude value, whereas the adjacent “northern Finland” arc (measured better) shows a *shortage* of about the same value; this collision discloses a significant deviation of the plumb line in the area of Torneå (from zenith southerly). Another example is a relatively *better appearance* of the Svanberg arc. Why so? Once Svanberg’s latitude values *may* have embraced observational errors, while fairly good quality of his terrestrial measurements is *proved*, one cannot help admitting *Svanberg’s luck* in obtaining the arc amplitude value (result of subtraction) wherein errors and geomorphologic deviations at either terminal seem to reduce each other considerably, leaving a relatively small residual only.

What about “*the length of one degree*” in Lapland? For the three measurements in view the quantities, accordingly, are equal to: 57437 (or 57438 with Maupertuis’s “55 023.47 toises”), 57196 and 57194 toises. More or less “right” values depending on the *Bm* (applying the above equation and using, e.g., A. Borsdorff’s ellipsoid) are these: 57216, 57216 and 57222 toises. The result is something of linear “errors” (+221, – 20, – 28 toises), speaking figuratively – “merged layers” which are, evidently, yet cruder estimates.

5. CONCLUSION

The 1st meridian arc measurement by Maupertuis. Mayor fault: the astronomical amplitude of the meridian arc joining the parallels of the terminal stations came to be shortened by ca – 9”. The resulting arc length value was ill chosen within the scatter of 105 m wide and its aggregate error came to be + 53 m (the estimate is uncertain within 10 m), or by ca 1/ 2000 per 107 km. The actual degree of inaccuracies during the observations: dozens of arcseconds in angles, arcminutes in astronomical orientation, a minor scale error by ca 1/ 10000.

The 2nd meridian arc measurement by Svanberg. No single evidence of express shortcomings of the geodetic part of the work; within the historic Lapland heptagon its degree of inaccuracy was single arcseconds in the angles and single metres in the length of the meridian arc closing the heptagon – quite near to the accuracy achieved afterwards by Selander with better instruments. Errors of astronomical observations have, apparently, distorted either of the terminal latitude values by few arcseconds, but the final angular amplitude fits the arc length value due to a favourable cross-compensation of the errors and accompanying geomorphologic deviations resulting from algebraic subtraction.

The 3rd meridian arc measurement by Selander. The length value of the geodesic joining the arc terminal stations has an aggregate shortening by ca 1/ 37000 per 324 km; in terms of the metre: by – 9 metres, or 2.7 times the respective probable error deduced by Struve. The length value of the meridian arc joining the parallels of the terminal stations, consequently, has been shortened by ca 1/ 50000 per 318 km; in terms of the metre: by – 6 metres, or within twofold respective probable error deduced by Struve. Rms errors of the angles exceed 1". Apart from geomorphologic influence, the astronomical amplitude value is accurate within 1" as the highest. Accuracy of Lindhagen's astronomical orientation fits corresponding Struve's estimate (rms error around $\pm 1''$).

This study was an experience of processing or just thinking over the appropriate data available to date. Some important details sure were missed, for, e.g., *luck* had to be involved; "luck" is often a substitute or label for positive factors not yet known. Anyway, no wonders have been discovered: the three successive astrogeodetic works demonstrate successively improving techniques of measurement.

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BIOGRAPHICAL NOTES

Born in 1947. Graduated in astronomy at the Faculty of Mathematics and Mechanics of the Leningrad University. Experience in satellite and astronomical observations, management. Headed volunteer expeditions aimed at preservation of artifacts of historic geodetic measurements. In 2003–2004 charged with compilation of the national documents for the FIG–UNESCO project “Struve Geodetic Arc”. Since 2004 – elected Secretary to the Board of the St.–Petersburg Society for Surveying & Mapping. Member of the Russian Geographical Society. Some 40 publications on historic geodetic measurements and artifacts in Russia.

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