GEOMETRIC ALGEBRAS AND CADASTRE

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Background

- **Almost all the current existing cadastral data models are based on Euclidean Geometry and Linear Algebra**
- **Do not have a uniform representation for spatial units with different dimensions**
- **The separation between geometric expression and algebraic computation in 3D spatial units representation**

Can we do something to make multidimensional cadastre more feasible?

- **Find a more direct expression method extend from the 2D cadastre to the 3D cadastre?**
- **Do the geometric operation and computation in a more algebraic and tuitive way**

Geometric Algebra

- **A tool to solve geometric problems in algebraic form**
- **Integrate the geometric and algebraic representations of spatial units**
- **Express the different primitives in a unified form.**

Basis of GA

- **Basic elements: vectors, bi-vectors and tri-vectors**
- **Basic operators: Outer product, inner product, geometric product**
- **Other concept: Meet, Dual, Inverse, Grades, Blade, Pseudo-scalar, etc.**

Basis of GA elements in 3D

• **1**	**scalar part**	**(grade 0)**
• **e1, e2, e3**	**vector part**	**(grade 1)**
• **e2^e3, e3^e1, e1^e2**	**bivector part**	**(grade 2)**
• **e1^e2^e3**	**trivector part**	**(grade 3)**

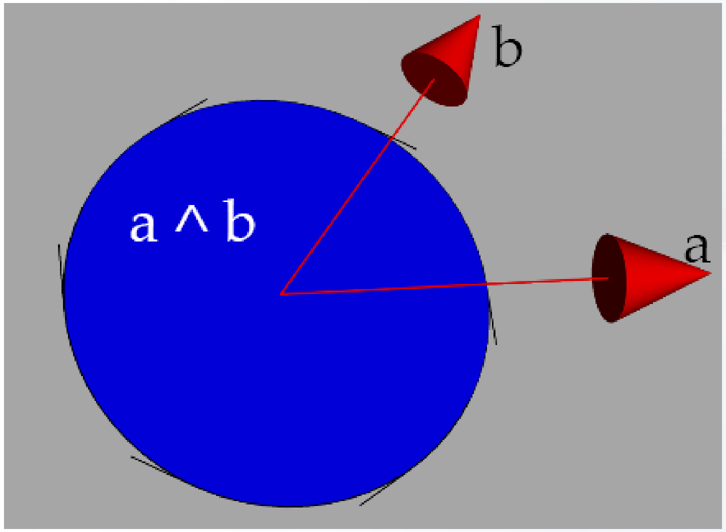
Outer product

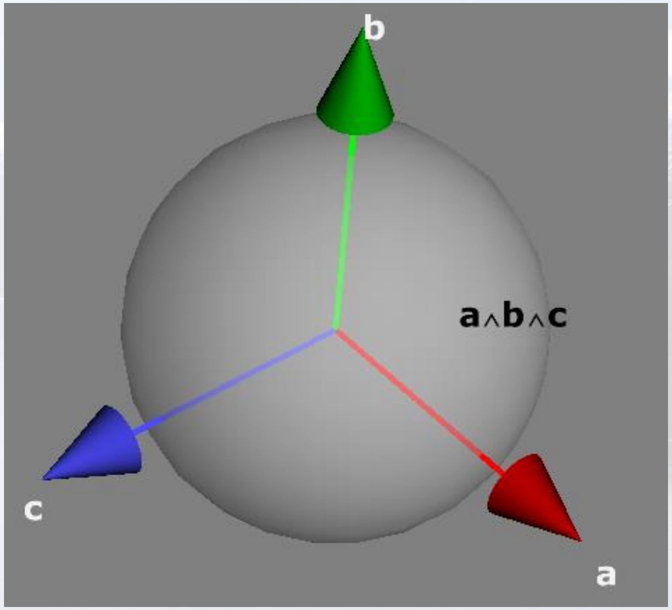
- **grade raising products**

- **combines elements of geometric algebra to form higher dimensional elements.(e.g. outer product of two vectors e1 and e2 produces a bivector e1^e2, and outer product of vector e3 and bivector e1^e2 produces a trivector e1^e2^e3). In generalize: grade(X^Y) = grade(X) + grade(Y).**

- **anti-symmetry: v ^ w = -w ^ v, so that v ^ v = 0**

Outer product





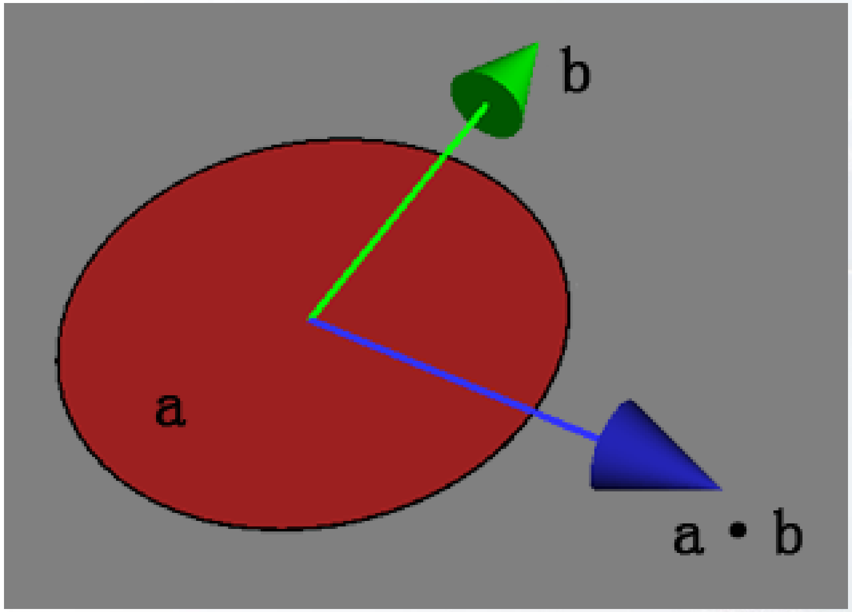
Inner product

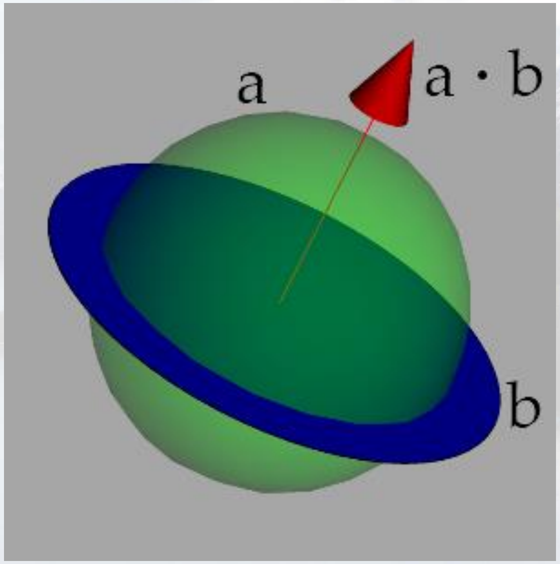
- **grade reducing products**

- **Inner product is generalization of dot product in different dimensions which reduce lower dimension from higher dimension (Inner product on vectors is the same with dot product, inner product of bivector and vector is to find the perpendicular of lower one in the higher one). In generalize: grade(X·Y) = grade(Y) - grade(X). And if grade(Y) < grade(X), we get zero.**

- **conditional symmetry : if grade(X) = grade(Y), then grade(X·Y) = 0. so, X·Y = Y·X.**

Inner product





Multivector

- **A multivector is a basic mathematical element that can simultaneously integrate multiple dimensional objects in geometric algebra.**
- **An example of multivector: e1+2*e1^e2+e3^e1.**

Conformal Geometric Algebra (CGA)

- A branch of geometric algebra.
- Traditional Euclidean 3D objects can be expressed in 5D conformal geometric space which is constructed by five basic vectors: e_1, e_2, e_3, e_0, e_∞.
- e_1, e_2, e_3 are the same as traditional vectors, e_0, e_∞ are two additional vectors denote original point and infinite point.
- The Grassmann structure corresponds to the hierarchical structure of dimensions in CGA

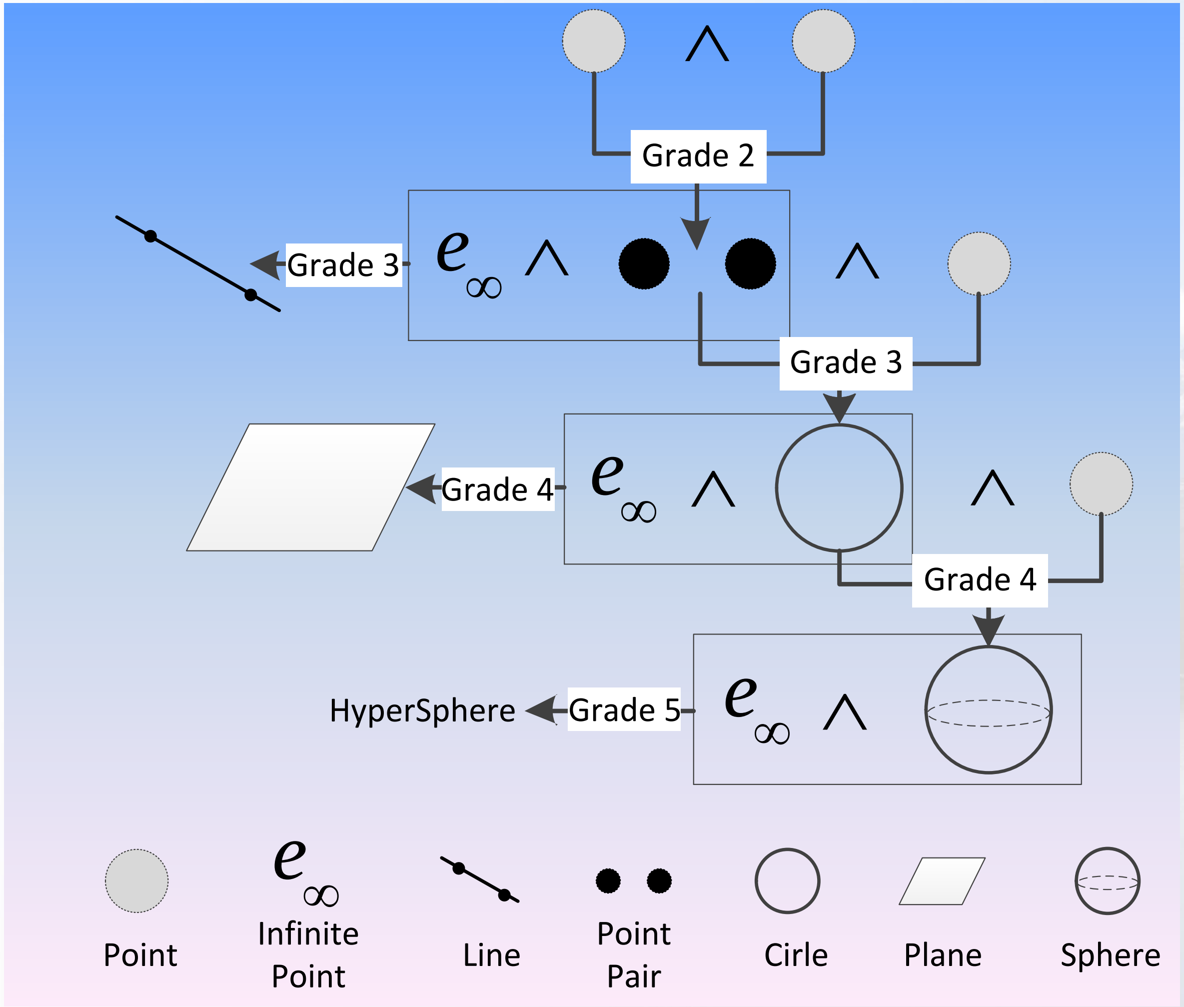
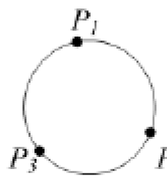
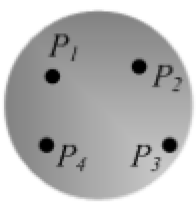

Grade 2
Grade 3
Grade 3
Grade 4
Grade 4
Grade 5
HyperSphere
Point
Infinite Point
Line
Point Pair
Cirle
Plane
Sphere

Table 1. Outer product expression of geometric objects in CGA.

Object	Drawing	Outer Production	Grade
Point		Null	1
Point pair		$PP = P_1 \wedge P_2$	2
Line		$L = P_1 \wedge P_2 \wedge e_\infty$	3
Circle		$C = P_1 \wedge P_2 \wedge P_3$	3
Plane		$P = P_1 \wedge P_2 \wedge P_3 \wedge e_\infty$	4
Sphere		$S = P_1 \wedge P_2 \wedge P_3 \wedge P_4$	4

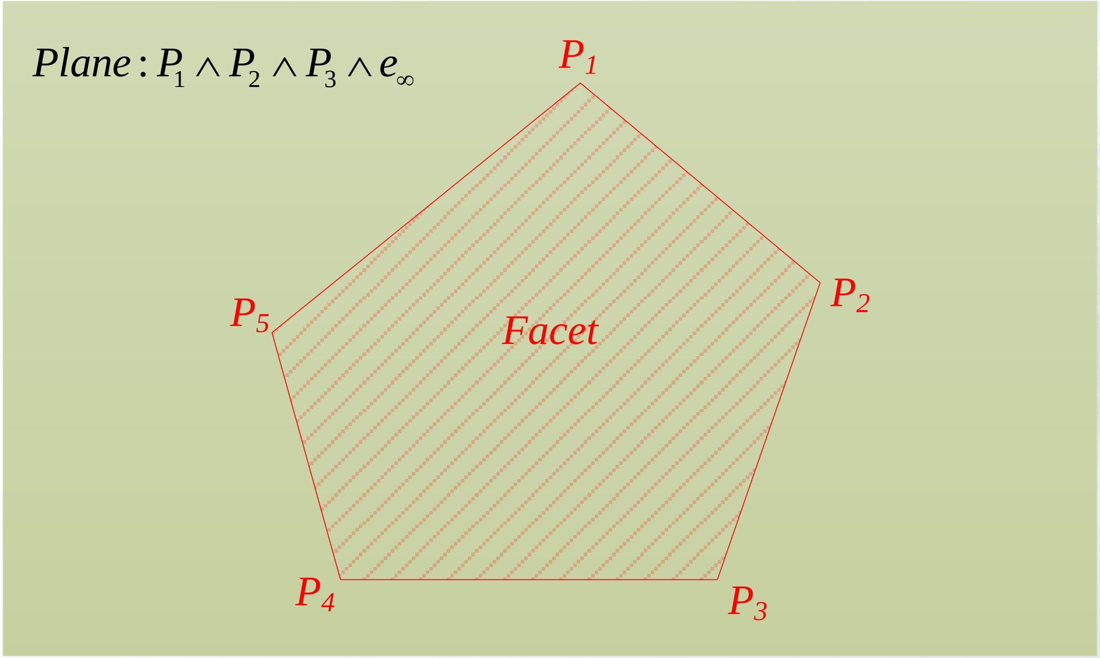
Cadastral primitives expression based on CGA

- Select points, boundary lines and boundary faces as the basic primitives for cadastral spatial units representation.

- So the three types of primitives' conformal expression can be defined as follows.

$$GeoPrim_{<k>} = \begin{pmatrix} P & k=1 \\ P_1 \wedge P_2 \wedge e_\infty & k=3 \\ P_1 \wedge P_2 \wedge P_3 \wedge e_\infty & k=4 \end{pmatrix} \{P_1, P_2, \ldots P_n\}$$

Cadastral primitives expression based on CGA



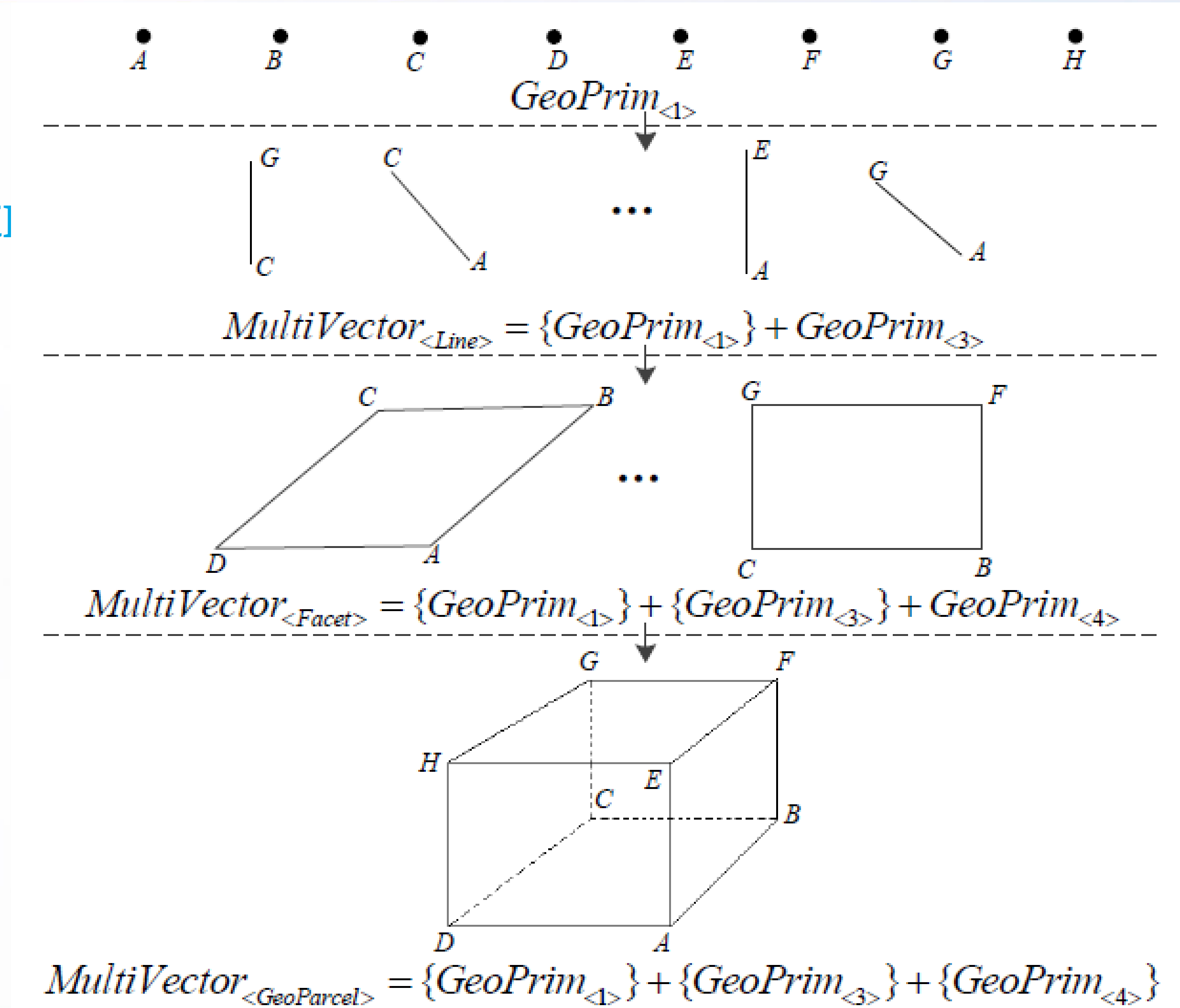
$$GeoPrim_{<4>} = (P_1 \wedge P_2 \wedge P_3 \wedge e_\infty)\{P_1, P_2, P_3, P_4, P_5\}$$

Cadastral spatial units expression based on CGA

- **Since cadastral spatial units maybe 2D or 3D objects that is composed by different dimensional primitives. The multivector structure in CGA is employed to organized and stored those components as follows.**

$$MultiVector_{<GeoParcel>} = \{GeoPrim_{<1>}\} + \{GeoPrim_{<3>}\} + \{GeoPrim_{<4>}\}$$

Cadast… A


A B C D E F G H
$GeoPrim_{<1>}$
G C
C A
E A
G A
$MultiVector_{<Line>} = \{GeoPrim_{<1>}\} + GeoPrim_{<3>}$
C B D A
G F C B
$MultiVector_{<Facet>} = \{GeoPrim_{<1>}\} + \{GeoPrim_{<3>}\} + GeoPrim_{<4>}$
G F H E C B D A
$MultiVector_{<GeoParcel>} = \{GeoPrim_{<1>}\} + \{GeoPrim_{<3>}\} + \{GeoPrim_{<4>}\}$

We have know that spatial units can be expressed in a unified form based on CGA.

Can we update them in a different way compare to traditional method?

CGA-based merging method

- **Spatial units merging method based on CGA include three steps:**
- **1. Get intersection results for two relative spatial units;**
- **2. If there is a common parts, then compute the results of Boundary operation for two relative spatial units to confirm the intersection of cadastral components;**
- **3. If the two spatial units need to merge, then update their multivector representation.**

Meet operation

- If A and B are two blades in $Cl_{4,1}$, the meet operator for A and B is defined as follow:
- $$M = Meet(A,B) = A \cap B = A^{*} \cdot B$$

$$M^{2} = \begin{cases} = 0 & no \;\; intersection \\ > 0 & intersection \;\; exists \end{cases}$$

- It should be noted that Meet operation only applied to blades rather than multivector.

Boundary operation

- **If *CA* and *CB* are two primitive components that contained in blades *A* and *B* in $Cl_{4,1}$ with the same dimensions, the boundary operator for them is defined as follow:**

$$Boundary(CA,CB)$$
$$= Boundary(Ptas,CB)\,\&\,Boundary(Ptbs,CA)$$
$$= (mPtas, nPtbs),(0,0)$$

- **Where *mPtas* and *nPtbs* are the boundary point sets that construct *CA* and *CB*, respectively. *mPtas* is the number of points belonging to *CA* that contained in *CB* and *nPtbs* is the number of points belonging to *CB* that contained in *CA*.**

Boundary operation

- **Suppose that *m* and *n* denote the total nunber points belonging to *CA* and *CB*, respectively. Then we can conclude as following:**
- **(1) if *mPtas=nPtbs=m=n* , this mean that *CA* and *CB* are the common parts which have to disappear in the process of cadastral objects merging;**
- **(2) if *0<mPtas=nPtbs<m=n* and the normal direction of *CA* and *CB* is the same, this mean that and are the two parts that possibly need to merge;**
- **(3) if *mPtas=nPtbs=0* , this mean that *CA* and *CB* are disjoint.**

Merging operation

- **Suppose *face*$_{<A>}$ and *face*$_{<B>}$ are two constructive faces in for cadastral A and B, their conformal representation are shown as following:**

$$Face_{<A>} = (P_{a_1} \wedge P_{a_2} \wedge P_{a_3} \wedge e_\infty)\{P_{a_1}, P_{a_2}, P_{a_3}, \cdots, P_w, P_x, \cdots, P_y, P_z, \cdots, P_{a_m}\}$$

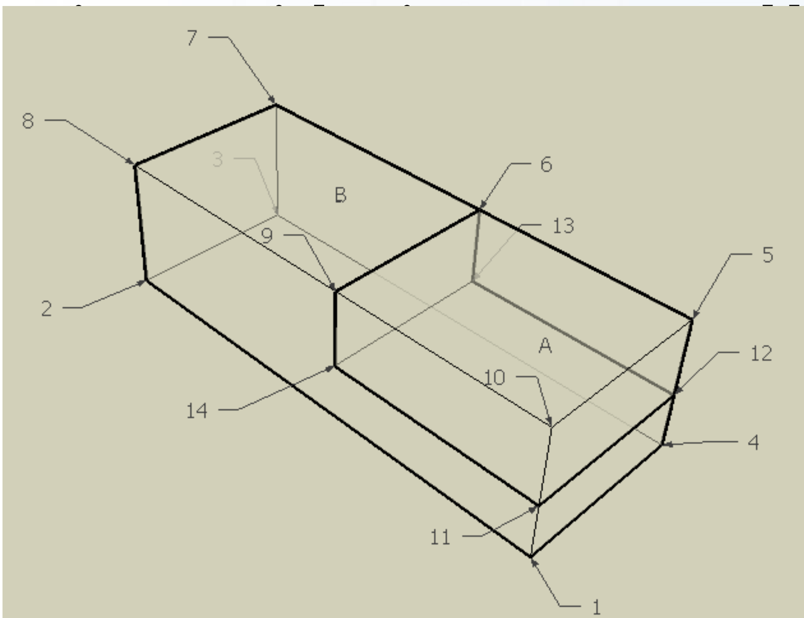
$$Facet_{<B>} = (P_{b_1} \wedge P_{b_2} \wedge P_{b_3} \wedge e_\infty)\{P_{b_1}, P_{b_2}, P_{b_3}, \cdots, P_s, P_y, \cdots, P_x, P_t, \cdots, P_{b_n}\}$$

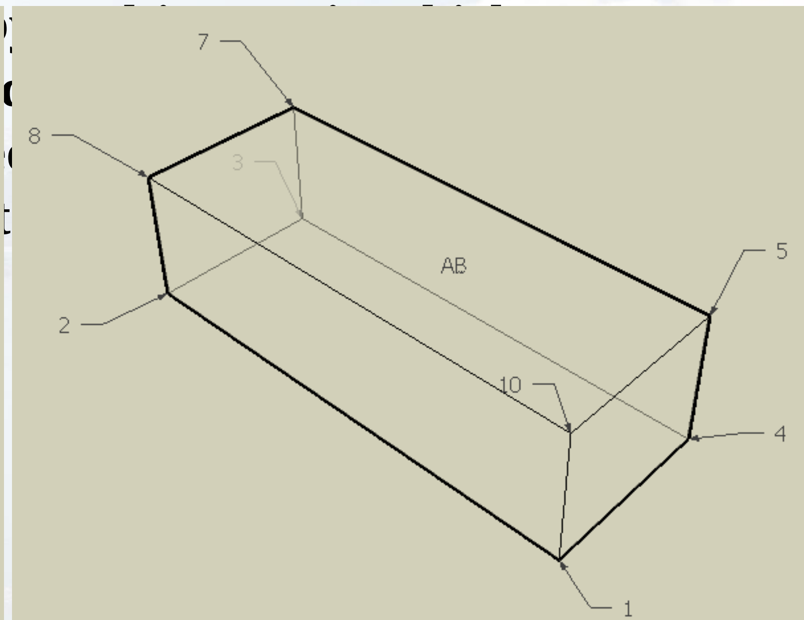
- **Then the result of combination is shown as follows:**

$$Facet_{<AB>} = (P_{b_1} \wedge P_{b_2} \wedge P_{b_3} \wedge e_\infty)\{P_{b_1}, ..., P_s, P_z, ..., P_{a_m}, P_{a_1}, ..., P_w, P_t, ...P_{b_n}\}$$

- **The point sets P_x, …, P_y *in face*$_{<A>}$ are same with the inversion of the point sets P_y, … , P_x in *face*$_{<B>}$.**

Merging result





Conclusion

- **Spatial units in all dimensions can be expressed using CGA in a unified form as an entity**

- **All components of spatial units are organized in the multivector structure**

- **Components with lower dimensions can be extracted from those with higher dimensions (from the boundary point sets)**

- **Spatial units updating by CGA operators can be realized in a more algebraic way**

References

- **Jiyi Zhang, Pengcheng Yin, Gang Li, Hehe Gu, Hua Zhao, Jianchun Fu (2016). 3D Cadastral Data Model Based on Conformal Geometry Algebra. ISPRS International Journal of Geo-Information, 5, 20, doi:10.3390/ijgi5020020.**
- **Hestenes's Site: http://modelingnts.la.asu.edu/**
- **Lounesto's Site: http://www.helsinki.fi/~lounesto/**
- **Cambridge Group: http://www.mrao.cam.ac.uk/~clifford/**