Univariate and Multivariate Models in M_{split} Estimation in the Context of Vertical Deformation Analysis

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Key words: M_{split} estimation, univariate model, multivariate model, deformation analysis

SUMMARY

The paper focuses on applying two approaches to M_{split} estimation in deformation analysis. The methods under consideration are the squared M_{split} estimation (SMS), which assumes the normality of the observation errors, and the absolute M_{split} estimation (AMS), which is based on L_1 norm condition. The main aim of the paper is to investigate such estimation types in the context of vertical displacement analysis with application of either of two models, namely the univariate and multivariate models. The Crude Monte Carlo simulations are the basis for obtaining estimation accuracies (both root-mean-square deviation, RMSD, and standard deviation, SD) and empirical systematic biases, additionally. The results are obtained for several different variants of point displacements. Here, it should be noted that accuracy of M_{split} estimates might depend on the values of such displacements. Generally, the univariate model in M_{split} estimation gives better accuracy if there are no gross errors in observation set. Considering such a model, one can say that SDs are lower for both SMS and AMS estimates. It is especially vivid for small displacements. This is very important from the practical point of view since small SDs result in smaller RMSDs. On the other hand, the multivariate model in M_{split} estimation might yield smaller systematic biases; however, smaller biases not always result in better accuracy. The variants which contain outliers show significant differences between applications of the univariate or multivariate models. One can say that these two approaches simply supplement each other. Generally, the outcomes confirm that the choice of the model in M_{split} estimation is important in deformation analysis because the appropriate approach allows to obtain superior accuracy. It is also confirmed that the accuracies and empirical biases of both M_{split} estimates depend not only on occurrence of gross errors in observation set but also on the values of the point displacements. The application of the univariate model is especially advisable when such displacements are relatively small. Finally, it is also noteworthy that AMS estimates give generally better results than SMS estimates.

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1. INTRODUCTION

Deformation analysis is an important task in surveying. Complexity of this problem causes development of different approaches to deformation analysis. One of the newest, still developing approach to displacement analysis is $M_{split(q)}$ estimation (e.g. Duchnowski and Wiśniewski 2011, 2014; Zienkiewicz 2015; Zienkiewicz and Baryła 2015). The assumption of such a method is that an observation set is unrecognized mixture of realizations of different *q* random variables (Wiśniewski 2009, 2010). Each observation might be assigned to *q* competitive functional models. The knowledge and experience of analyst and/or particular estimation problem are the basis of assuming number of competitive models (Zienkiewicz 2020).

The most common case of $M_{split(q)}$ estimation is M_{split} estimation which concerns two competitive functional models, hence two competitive parameters or parameter vectors. In the case of vertical displacement analysis, observation set might be a mixture of observations from two different measurement epochs. Then M_{split} estimation allows us to obtain parameters from these two epochs in automatic way; such method does not require to separate observations from different epochs. The paper focuses on applying two approaches to M_{split} estimation in vertical deformation analysis. The first method under consideration is the squared M_{split} estimation (SMS), which assumes the normality of the observation errors (Wiśniewski 2009, 2010). The second one is the absolute M_{split} estimation (AMS), which is based on L_1 norm condition (Wyszkowska and Duchnowski 2019, 2020). Up to now, the multivariate model of M_{split} estimation was usually considered in deformation analysis. The main aim of the paper is to investigate both approaches to M_{split} estimation in the context of chosen levelling network with application of mentioned the multivariate model and also the univariate model. The paper is focused on the estimation accuracies of vertical point displacements (both root-mean-square deviation, RMSD, and standard deviation, SD) and the empirical systematic biases, additionally. All empirical analyses are based on Crude Monte Carlo simulations.

2. THEORETICAL FOUNDATIONS

The conventional functional model in M_{split} estimation can be split into two competitive models (Wiśniewski 2009):

$$\mathbf{y} = \boldsymbol{\theta} + \mathbf{v} = \mathbf{AX} + \mathbf{v} \overset{\text{split}}{\Rightarrow} \begin{matrix} \mathbf{y} = \boldsymbol{\theta}_{(1)} + \mathbf{v}_{(1)} = \mathbf{AX}_{(1)} + \mathbf{v}_{(1)} \\ \mathbf{y} = \boldsymbol{\theta}_{(2)} + \mathbf{v}_{(2)} = \mathbf{AX}_{(2)} + \mathbf{v}_{(2)} \end{matrix} \tag{1}$$

where: $\mathbf{y}$ – vector of observations, $\boldsymbol{\theta}$ – vector of location parameters, $\mathbf{v}$ – vector of measurement errors, $\mathbf{A}$ – matrix of coefficients, $\mathbf{X}$ – vector of parameters. The split of the functional model concerns the same observation set $\mathbf{y}$. Generally, the observation set in M_{split} estimation is an unknown mixture of realizations of two random variables which differ from one another at least in the location parameters. The assignment of particular observation to a respective parameter version is automatic during iterative adjustment. In other words, there is no prior information about the division of the observations into two aggregations.
The optimization problem of M_{split} estimation is written as minimization of the objective function $\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})$ (Wiśniewski, 2009; 2010):

$$\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})=\sum_{i=1}^{n}\rho_{(1)}(v_{i(1)},v_{i(2)})\rho_{(2)}(v_{i(1)},v_{i(2)})=\min_{\mathbf{X}_{(1)},\mathbf{X}_{(2)}} \tag{2}$$

where: ρ – arbitrary function which defines the objective function. The influence functions $\psi_{(1)}(v_{(1)},v_{(2)})$ and $\psi_{(2)}(v_{(1)},v_{(2)})$, weight functions $w_{(1)}(v_{(1)},v_{(2)})$ and $w_{(2)}(v_{(1)},v_{(2)})$ are as follows (e.g., Wiśniewski, 2009):

$$\begin{aligned}\psi_{(1)}(v_{(1)},v_{(2)})&=\frac{\partial\rho_{(1)}(v_{i(1)},v_{i(2)})\rho_{(2)}(v_{i(1)},v_{i(2)})}{\partial v_{(1)}}=\rho_{(2)}(v_{i(1)},v_{i(2)})\frac{\partial\rho_{(1)}(v_{i(1)},v_{i(2)})}{\partial v_{(1)}}\\ \psi_{(2)}(v_{(1)},v_{(2)})&=\frac{\partial\rho_{(1)}(v_{i(1)},v_{i(2)})\rho_{(2)}(v_{i(1)},v_{i(2)})}{\partial v_{(2)}}=\rho_{(1)}(v_{i(1)},v_{i(2)})\frac{\partial\rho_{(2)}(v_{i(1)},v_{i(2)})}{\partial v_{(2)}}\end{aligned} \tag{3}$$

$$w_{(1)}(v_{(1)},v_{(2)})=\frac{\psi_{(1)}(v_{(1)},v_{(2)})}{2v_{(1)}}\qquad w_{(2)}(v_{(1)},v_{(2)})=\frac{\psi_{(2)}(v_{(1)},v_{(2)})}{2v_{(2)}} \tag{4}$$

The first approach to M_{split} estimation is the squared M_{split} estimation (SMS), which assumes the normality of the observation errors (Wiśniewski 2009, 2010). The objective function of SMS estimation is defined as (Wiśniewski 2009):

$$\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})=\sum_{i=1}^{n}\rho_{(1)}(v_{i(1)},v_{i(2)})\rho_{(2)}(v_{i(1)},v_{i(2)})=\sum_{i=1}^{n}v_{i(1)}^{2}v_{i(2)}^{2} \tag{5}$$

The respective influence functions and weight functions have the following forms:

$$\psi_{(1)}(v_{(1)},v_{(2)})=2v_{(1)}v_{(2)}^{2}\qquad \psi_{(2)}(v_{(1)},v_{(2)})=2v_{(1)}^{2}v_{(2)} \tag{6}$$

$$w_{(1)}(v_{(1)},v_{(2)})=\frac{\psi_{(1)}(v_{(1)},v_{(2)})}{2v_{(1)}}=v_{(2)}^{2}\qquad w_{(2)}(v_{(1)},v_{(2)})=\frac{\psi_{(2)}(v_{(1)},v_{(2)})}{2v_{(2)}}=v_{(1)}^{2} \tag{7}$$

M_{split} estimation uses an iterative process based on the Newton process (Wiśniewski 2009). In the case of SMS method the starting point of the iterative process $\hat{\mathbf{X}}^{0}$ is usually the least squares estimate (LS), namely $\hat{\mathbf{X}}_{LS}$ (Wiśniewski 2009, 2010). However, that method might have different starting points like (Wyszkowska and Duchnowski 2020):

$$\hat{\mathbf{X}}^{0}=\hat{\mathbf{X}}_{LS}+\boldsymbol{\Delta} \tag{8}$$

where: $\boldsymbol{\Delta}$ – vector of assumed positive values. Generally, the iterative process for SMS estimation can be written as follows (e.g., Wiśniewski 2009; Wyszkowska and Duchnowski 2019):

$$\begin{aligned}\mathbf{X}_{(1)}^{j} &= \mathbf{X}_{(1)}^{j-1} + d\mathbf{X}_{(1)}^{j} = \mathbf{X}_{(1)}^{j-1} - \left[\mathbf{H}_{(1)}\left(\mathbf{X}_{(1)}^{j-1}, \mathbf{X}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(1)}\left(\mathbf{X}_{(1)}^{j-1}, \mathbf{X}_{(2)}^{j-1}\right) \\ \mathbf{X}_{(2)}^{j} &= \mathbf{X}_{(2)}^{j-1} + d\mathbf{X}_{(2)}^{j} = \mathbf{X}_{(2)}^{j-1} - \left[\mathbf{H}_{(2)}\left(\mathbf{X}_{(1)}^{j}, \mathbf{X}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(2)}\left(\mathbf{X}_{(1)}^{j}, \mathbf{X}_{(2)}^{j-1}\right)\end{aligned} \tag{9}$$

where: $d\mathbf{X}$ – increment to parameter, $\mathbf{H}$ – Hessian matrix, $\mathbf{g}$ – gradient, where:

$$\begin{aligned}&\begin{cases}\mathbf{H}_{(1)}\left(\mathbf{X}_{(1)}^{j-1}, \mathbf{X}_{(2)}^{j-1}\right) = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \\ \mathbf{g}_{(1)}\left(\mathbf{X}_{(1)}^{j-1}, \mathbf{X}_{(2)}^{j-1}\right) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(1)}^{j-1} \\ \mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) = diag\left[w_{(1)}(v_{1(1)}^{j-1}, v_{1(2)}^{j-1}), \ldots, w_{(1)}(v_{n(1)}^{j-1}, v_{n(2)}^{j-1})\right]\end{cases} \\ &\begin{cases}\mathbf{H}_{(2)}\left(\mathbf{X}_{(1)}^{j}, \mathbf{X}_{(2)}^{j-1}\right) = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \\ \mathbf{g}_{(2)}\left(\mathbf{X}_{(1)}^{j}, \mathbf{X}_{(2)}^{j-1}\right) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(2)}^{j-1} \\ \mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right) = diag\left[w_{(2)}(v_{1(1)}^{j}, v_{1(2)}^{j-1}), \ldots, w_{(2)}(v_{n(1)}^{j}, v_{n(2)}^{j-1})\right]\end{cases}\end{aligned} \tag{10}$$

The iterative process is finished for such $j = k$, for which both gradients equals zero and hence $\hat{\mathbf{X}}_{(1)} = \mathbf{X}_{(1)}^{k} = \mathbf{X}_{(1)}^{k-1}$, $\hat{\mathbf{X}}_{(2)} = \mathbf{X}_{(2)}^{k} = \mathbf{X}_{(2)}^{k-1}$ or at least both gradients are close enough to zero and $\left|d\mathbf{X}_{(1)}^{k}\right| < \varepsilon$ and $\left|d\mathbf{X}_{(2)}^{k}\right| < \varepsilon$, where ε – assumed small positive number.

The second approach to M_{split} estimation is the absolute M_{split} estimation (AMS estimation) (Wyszkowska and Duchnowski 2019) which is based on L_1 norm condition (e.g., Marshall and Bethel 1996; Baselga and García-Asenjo 2008). The objective function of AMS estimation is described in the following form:

$$\varphi(\mathbf{X}_{(1)}, \mathbf{X}_{(2)}) = \sum_{i=1}^{n} \rho_{(1)}(v_{i(1)}, v_{i(2)})\rho_{(2)}(v_{i(1)}, v_{i(2)}) = \sum_{i=1}^{n} \left|v_{i(1)}\right|\left|v_{i(2)}\right| \tag{11}$$

The respective influence functions and weight functions are following:

$$\psi_{(1)}(v_{(1)}, v_{(2)}) = \begin{cases}-\left|v_{(2)}\right| & \text{for } v_{(1)} < 0 \\ \left|v_{(2)}\right| & \text{for } v_{(1)} > 0\end{cases} \qquad \psi_{(2)}(v_{(1)}, v_{(2)}) = \begin{cases}-\left|v_{(1)}\right| & \text{for } v_{(2)} < 0 \\ \left|v_{(1)}\right| & \text{for } v_{(2)} > 0\end{cases} \tag{12}$$

$$w_{(1)}(v_{(1)}, v_{(2)}) = \begin{cases}-\dfrac{\left|v_{(2)}\right|}{2v_{(1)}} & \text{for } v_{(1)} < 0 \\ \dfrac{\left|v_{(2)}\right|}{2v_{(1)}} & \text{for } v_{(1)} > 0\end{cases} \qquad w_{(2)}(v_{(1)}, v_{(2)}) = \begin{cases}-\dfrac{\left|v_{(1)}\right|}{2v_{(2)}} & \text{for } v_{(2)} < 0 \\ \dfrac{\left|v_{(1)}\right|}{2v_{(2)}} & \text{for } v_{(2)} > 0\end{cases} \tag{13}$$

The iterative process of AMS estimation is different from this one of SMS method because of the lack of mutual cross-weighting (Wyszkowska and Duchnowski 2019). Another difference for AMS estimation is that such a method requires two different starting points $\hat{\mathbf{X}}^0_{(1)}$ and $\hat{\mathbf{X}}^0_{(2)}$ (or more generally q starting points for q competitive functional models). This is necessary since assumption of the same two starting points for the both parameter vectors causes the failure at starting an iterative process. The basic solution of the starting points for AMS estimation is as follows (Wyszkowska and Duchnowski 2019, 2020):

$$\begin{aligned}\hat{\mathbf{X}}^0_{(1)} &= \hat{\mathbf{X}}_{LS} + \mathbf{\Delta}\\ \hat{\mathbf{X}}^0_{(2)} &= \hat{\mathbf{X}}_{LS} - \mathbf{\Delta}\end{aligned} \qquad (14)$$

The iterative process of AMS estimation may be described as (Wyszkowska and Duchnowski 2019, 2020):

$$\begin{aligned}\mathbf{X}^j_{(1)} &= \mathbf{X}^{j-1}_{(1)} + d\mathbf{X}^j_{(1)}\\ d\mathbf{X}^j_{(1)} &= \left[\mathbf{H}_{(1)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right)\right]^{-1} \mathbf{g}_{(1)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right)\\ \mathbf{X}^j_{(2)} &= \mathbf{X}^{j-1}_{(2)} + d\mathbf{X}^j_{(2)}\\ d\mathbf{X}^j_{(2)} &= \left[\mathbf{H}_{(2)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right)\right]^{-1} \mathbf{g}_{(2)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right)\end{aligned} \qquad (15)$$

where:

$$\begin{aligned}&\begin{cases}\mathbf{H}_{(1)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right) = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right)\mathbf{A}\\ \mathbf{g}_{(1)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(1)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right)\mathbf{v}^{j-1}_{(1)}\\ \mathbf{w}_{(1)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right) = diag\left[w^*_{(1)}(v^{j-1}_{1(1)}, v^{j-1}_{1(2)}), \ldots, w^*_{(1)}(v^{j-1}_{n(1)}, v^{j-1}_{n(2)})\right]\end{cases}\\ &\begin{cases}\mathbf{H}_{(2)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right) = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right)\mathbf{A}\\ \mathbf{g}_{(2)}\left(\mathbf{X}^{j-1}_{(1)}, \mathbf{X}^{j-1}_{(2)}\right) = -2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{(2)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right)\mathbf{v}^{j-1}_{(2)}\\ \mathbf{w}_{(2)}\left(\mathbf{v}^{j-1}_{(1)}, \mathbf{v}^{j-1}_{(2)}\right) = diag\left[w^*_{(2)}(v^{j-1}_{1(1)}, v^{j-1}_{1(2)}), \ldots, w^*_{(2)}(v^{j-1}_{n(1)}, v^{j-1}_{n(2)})\right]\end{cases}\end{aligned} \qquad (16)$$

where:

$$w^*_{(1)}(v_{(1)}, v_{(2)}) = \begin{cases}\dfrac{|v_{(2)}|}{2|v_{(1)}|} & \text{for } |v_{(1)}| \geq d\\ \dfrac{|v_{(2)}|}{2d} & \text{for } |v_{(1)}| < d\end{cases} \qquad w^*_{(2)}(v_{(1)}, v_{(2)}) = \begin{cases}\dfrac{|v_{(1)}|}{2|v_{(2)}|} & \text{for } |v_{(2)}| \geq d\\ \dfrac{|v_{(1)}|}{2d} & \text{for } |v_{(2)}| < d\end{cases} \qquad (17)$$

are modifications of weight functions from Eq. (13), d – small assumed positive constant (e.g. d = 0.001). These modifications of weight functions are necessary because of possible singularity resulting from the weight functions of Eq. (13) which are not defined for measurement errors equal to zero. The condition of ending iterative process of AMS estimation is the same as for SMS estimation.

3. EMPIRICAL TESTS

The empirical tests are based on the following simulated levelling network (Fig. 1). Such a network consists of two reference points P_1, P_2 and three object points A, B, C. All height differences h_i are measured at two epochs.

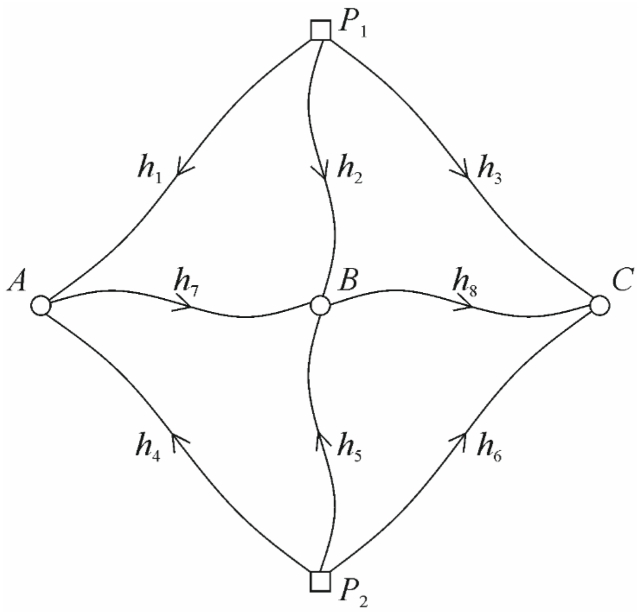

Fig. 1. Simulated levelling network

In the case of M_{split} estimation in vertical displacement analysis, observation set might be a mixture of observations from two different measurement epochs. In this paper one applies either of two models, namely the univariate or multivariate models. In the case of the multivariate model, we have following matrix of coefficients **A**, vector of observations **y**, vectors of parameters $\mathbf{X}_{(1)}$, $\mathbf{X}_{(2)}$ and difference of parameters $\mathbf{\Delta X}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T \tag{18}$$

$$\mathbf{y} = \begin{bmatrix} h_1^I & h_1^{II} & h_2^I & h_2^{II} & h_3^I & h_3^{II} & h_4^I & h_4^{II} & h_5^I & h_5^{II} & h_6^I & h_6^{II} & h_7^I & h_7^{II} & h_8^I & h_8^{II} \end{bmatrix}^T \tag{19}$$

$$\mathbf{X}_{(1)} = \begin{bmatrix} H_A^I \\ H_B^I \\ H_C^I \end{bmatrix} \qquad \mathbf{X}_{(2)} = \begin{bmatrix} H_A^{II} \\ H_B^{II} \\ H_C^{II} \end{bmatrix} \tag{20}$$

$$\mathbf{\Delta X} = \mathbf{X}_{(2)} - \mathbf{X}_{(1)} = \begin{bmatrix} H_A^{II} \\ H_B^{II} \\ H_C^{II} \end{bmatrix} - \begin{bmatrix} H_A^I \\ H_B^I \\ H_C^I \end{bmatrix} = \begin{bmatrix} \Delta H_A \\ \Delta H_B \\ \Delta H_C \end{bmatrix} \tag{21}$$

The univariate model is not commonly used in the context of M_{split} estimation in vertical displacement analysis. For that model we have following matrices of coefficients $\mathbf{A}_A$, $\mathbf{A}_B$, $\mathbf{A}_C$ for object points A, B, C, vectors of observations $\mathbf{y}_A$, $\mathbf{y}_B$, $\mathbf{y}_C$, matrices of weights $\mathbf{P}_A$, $\mathbf{P}_B$, $\mathbf{P}_C$ and then parameters $X_{A(1)}$, $X_{B(1)}$, $X_{C(1)}$ and $X_{A(2)}$, $X_{B(2)}$, $X_{C(2)}$:

$$\begin{gathered} \mathbf{A}_A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \\ \mathbf{A}_B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \\ \mathbf{A}_C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T \end{gathered} \tag{22}$$

$$\begin{gathered} \mathbf{y}_A = \begin{bmatrix} h_1^I + H_{P_1} & h_1^{II} + H_{P_1} & h_4^I + H_{P_2} & h_4^{II} + H_{P_2} & h_2^I - h_7^I + H_{P_1} & h_2^{II} - h_7^{II} + H_{P_1} \end{bmatrix}^T \\ \mathbf{y}_B = \begin{bmatrix} h_2^I + H_{P_1} & h_2^{II} + H_{P_1} & h_5^I + H_{P_2} & h_5^{II} + H_{P_2} & h_1^I + h_7^I + H_{P_1} & h_1^{II} + h_7^{II} + H_{P_1} & h_6^I - h_8^I + H_{P_2} & h_6^{II} - h_8^{II} + H_{P_2} \end{bmatrix}^T \\ \mathbf{y}_C = \begin{bmatrix} h_3^I + H_{P_1} & h_3^{II} + H_{P_1} & h_6^I + H_{P_2} & h_6^{II} + H_{P_2} & h_5^I + h_8^I + H_{P_2} & h_5^{II} + h_8^{II} + H_{P_2} \end{bmatrix}^T \end{gathered} \tag{23}$$

$$\mathbf{P}_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \mathbf{P}_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \mathbf{P}_C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{24}$$

$$
\begin{array}{ll}
X_{A(1)} = H_A^I & X_{A(2)} = H_A^{II} \\
X_{B(1)} = H_B^I & X_{B(2)} = H_B^{II} \\
X_{C(1)} = H_C^I & X_{C(2)} = H_C^{II}
\end{array}
\tag{25}
$$

where: H_{P_1}, H_{P_2} – heights of reference points P_1 and P_2, H^I – height at the first measurement epoch, H^{II} – height at the second measurement epoch. The vertical displacements of object points *A*, *B*, *C* can be computed by following formula:

$$
\begin{array}{l}
\Delta X_A = X_{A(2)} - X_{A(1)} = H_A^{II} - H_A^I = \Delta H_A \\
\Delta X_B = X_{B(2)} - X_{B(1)} = H_B^{II} - H_B^I = \Delta H_B \\
\Delta X_C = X_{C(2)} - X_{C(1)} = H_C^{II} - H_C^I = \Delta H_C
\end{array}
\tag{26}
$$

where: ΔX – difference of parameters, ΔH – vertical point displacement.
In this paper empirical tests are executed in Mathcad 15.0 on the basis of Crude Monte Carlo (MC) simulations, which are applied in solving many geodetic or surveying issues (e.g. Xu 2005; Duchnowski, Wiśniewski 2014; Wyszkowska 2017). In these tests there is an assumption that all height differences are independent and their errors have normal distributions $v_i \sim N(0,\sigma^2)$, where standard deviation $\sigma = 1\,\text{mm}$. We carry out 1000 simulations for each empirical test. These tests present the accuracies of M_{split} estimates, namely both root-mean-square deviation (RMSD) and empirical standard deviation (SD) and empirical systematic biases of M_{split} estimates (e.g., Duchnowski, Wiśniewski 2014, 2017):

$$
\text{RMSD}(\hat{X}) = \sqrt{\sum_{i=1}^{n} \frac{\left(\hat{X}_i^{MC} - X\right)^2}{n}} \tag{27}
$$

$$
\text{SD}(\hat{X}) = \sqrt{\sum_{i=1}^{n} \frac{\left(\hat{X}_i^{MC} - \bar{X}^{MC}\right)^2}{n}} \tag{28}
$$

$$
\text{Bias}(\hat{X}) = \bar{X}^{MC} - X \tag{29}
$$

where: $\hat{X}_i^{MC}$ – estimated value at the *i*th Monte Carlo simulation, $\bar{X}^{MC}$ – mean value of the parameter from Monte Carlo simulations, *n* – number of simulations, *X* – theoretical value of the estimated parameter. One should mention about the starting points in our empirical tests in iterative processes of both types of M_{split} estimations. For SMS estimation, $\mathbf{X}_{(1)}^0 = \hat{\mathbf{X}}_{LS}$ is the most often used starting point; however, such a starting point might lead to wrong solutions in the multivariate model (Wyszkowska and Duchnowski 2019, 2020). Hence, in this paper $\mathbf{X}_{(1)}^0 = \hat{\mathbf{X}}_{LS} + 10\,\text{mm}$ for SMS estimation and $\mathbf{X}_{(1)}^0 = \hat{\mathbf{X}}_{LS} - 10\,\text{mm}$ and $\mathbf{X}_{(2)}^0 = \hat{\mathbf{X}}_{LS} + 10\,\text{mm}$ for AMS estimation. Another noteworthy issue is that both iterative processes end when $\left|d\mathbf{X}_{(1)}^k\right| < 0.001\,\text{mm}$ and $\left|d\mathbf{X}_{(2)}^k\right| < 0.001\,\text{mm}$.

Without loss of generality, one can assume that $H_{P_1}^{\mathrm{I}} = H_{P_1}^{\mathrm{II}} = 0\ \mathrm{mm}$ and $H_{P_2}^{\mathrm{I}} = H_{P_2}^{\mathrm{II}} = 0\ \mathrm{mm}$ at both measurement epochs; in other words the reference points are stable. In the case of the object points, we assume that $H_A^{\mathrm{I}} = 0\ \mathrm{mm}$, $H_B^{\mathrm{I}} = 0\ \mathrm{mm}$, $H_C^{\mathrm{I}} = 0\ \mathrm{mm}$ at the first measurement epoch. Additionally, let H_A^{II} vary within interval $[0\ \mathrm{mm}, 50\ \mathrm{mm}]$ and let us analyse some variants of different values of H_B^{II}, and H_C^{II} and hence different magnitudes of the vertical displacements ΔH_B, ΔH_C. Additionally, these variants differ from each other in the fact that the observation set may contain one gross error equal to 5 mm which affects h_1^{II}.

3.1 Variant I for $H_B^{\mathrm{II}} = 0\ \mathrm{mm}$, $H_C^{\mathrm{II}} = 0\ \mathrm{mm}$

First, let us analyse the variant with zero magnitude of the vertical displacements of the object points *B* and *C*. Fig. 2 presents RMSDs, SDs and biases obtained for M_{split} estimates.

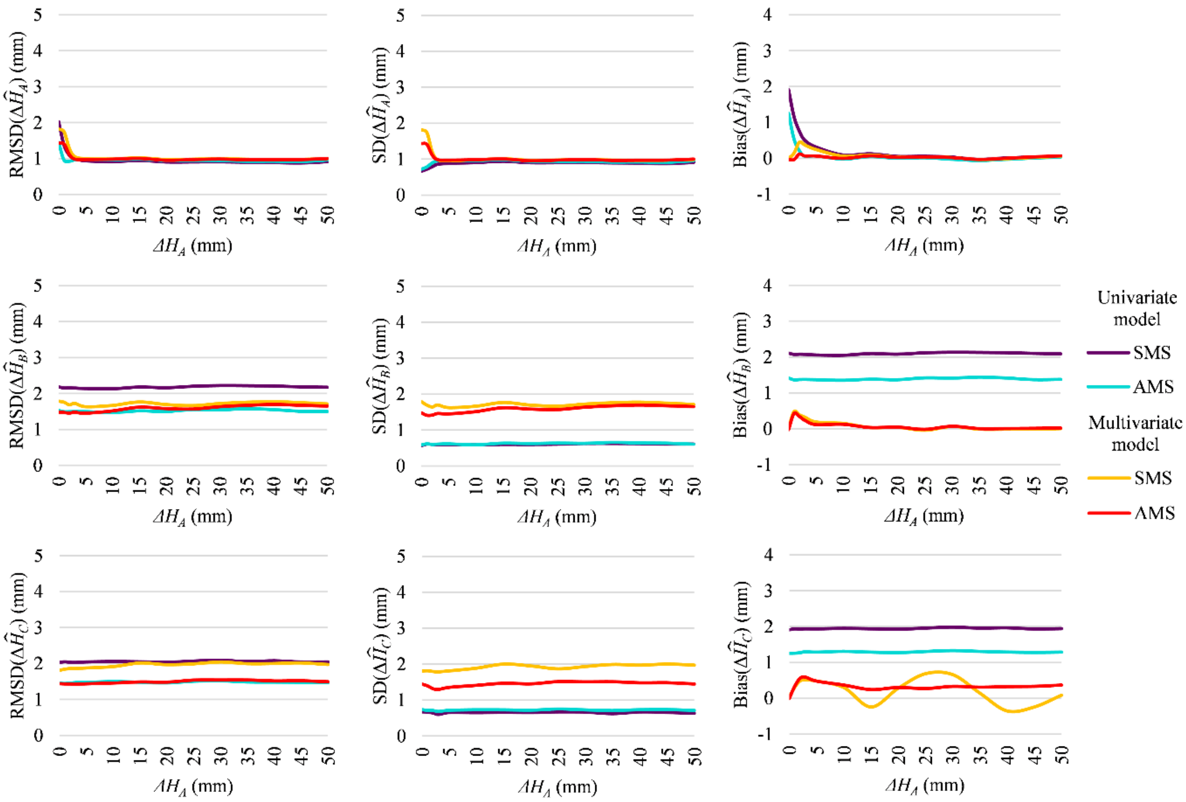

Fig. 2. RMSDs, SDs and biases of point displacements for different values of ΔH_A; where $\Delta H_B = 0$ mm, $\Delta H_C = 0$ mm (Variant I)

For growing ΔH_A all results concerning the object point *A* are similar to each other. However, if the magnitude of ΔH_A are quite small, then there are some disturbances for $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$, $\mathrm{SD}\left(\Delta\hat{H}_A\right)$ and $\mathrm{Bias}\left(\Delta\hat{H}_A\right)$. The probable cause is that there might be some problems with assigning observations to the appropriate measurement epochs when the parameter values are close to each other at both epochs. Let us consider the object points *B* and *C*. RMSDs are better for AMS estimation and these results are similar to one another for the univariate and multivariate models. What is more, SDs for the univariate model are much smaller than RMSDs. However, RMSDs and SDs of the multivariate model are similar to one another. Additionally, there are noticeable discrepancies of biases between both models considered; superior values of biases are obtained for the multivariate model. Note that in the context of such a model applied in SMS estimation, ΔH_A influences the values of $\mathrm{Bias}\left(\Delta\hat{H}_C\right)$ and the smallest biases are acquired for ΔH_A approximate to 30 mm.

3.2 Variant II for $H_B^{\mathrm{II}} = 0\,\mathrm{mm}$, $H_C^{\mathrm{II}} = 0\,\mathrm{mm}$ and $h_1^{\mathrm{II}} + 5\,\mathrm{mm}$

In comparison to Variant I, in Variant II h_1^{II} is affected by gross error equal to 5 mm. Fig. 3 shows respective RMSDs, SDs and biases of point displacements for different values of ΔH_A.

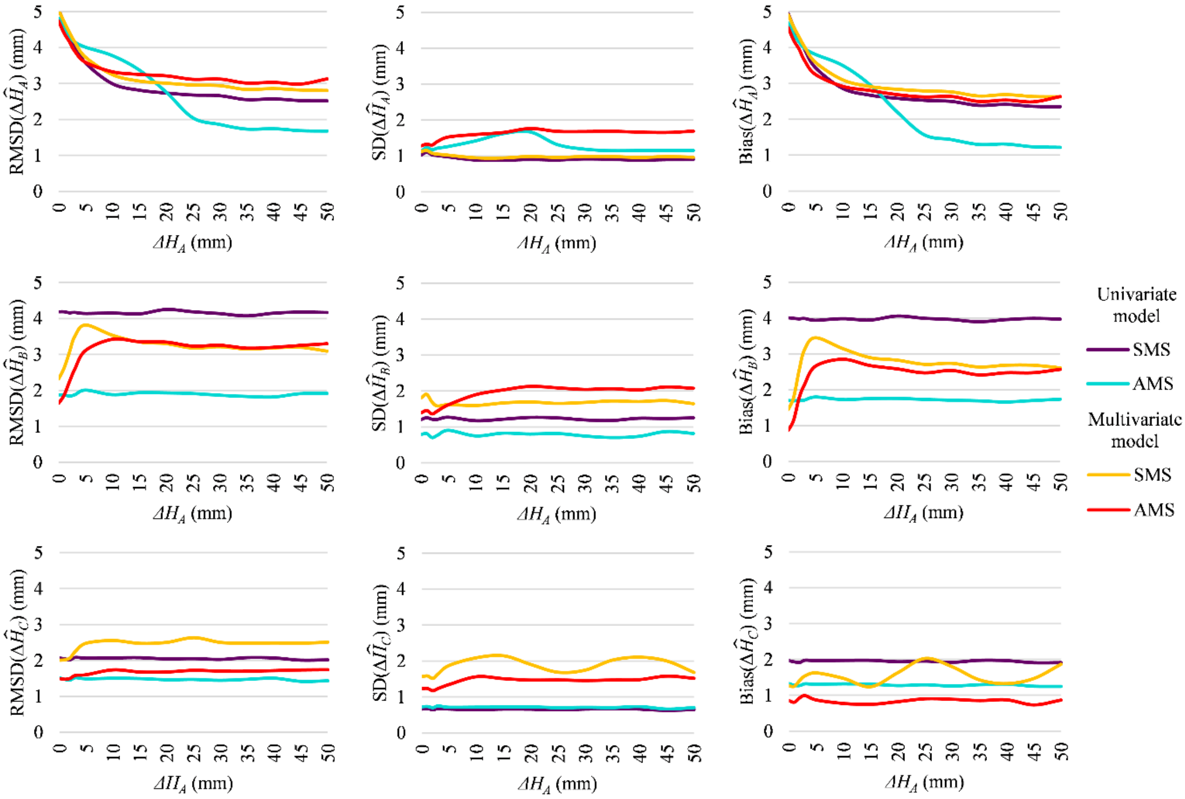

Fig. 3. RMSDs, SDs and biases of point displacements for different values of ΔH_A; where $\Delta H_B = 0$ mm, $\Delta H_C = 0$ mm and $h_1^{\mathrm{II}} + 5\,\mathrm{mm}$ (Variant II)

Generally, the superior results of RMSDs and SDs are obtained for the univariate model. The exceptions are $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ for AMS method for $\Delta H_A < 20\,\mathrm{mm}$ and $\mathrm{RMSD}\left(\Delta\hat{H}_B\right)$ for

SMS method. Another interesting issue is that AMS estimation gives much better RMSDs than SMS estimation, which is especially vivid for $\Delta\hat{H}_B$ for the univariate model. Also, such a model gives smaller SDs. Moreover, the shapes of biases for object points *A* and *B* are close to the shapes of respective RMSDs. What is more, the results presented here show that both M_{split} estimates of the displacements ΔH_A and ΔH_B are strongly influenced by gross error. Only AMS estimation for the univariate model gives quite similar results for $\Delta\hat{H}_B$ to these ones in Variant I. It is also noteworthy that the most affected estimates are these ones which concern the points which are the closest to the outlier, namely $\Delta\hat{H}_A$.

3.3 Variant III for $H_B^{II} = 20$ mm, $H_C^{II} = 50$ mm

Let us assume other values of the heights of the object points *B* and *C* at second measurement epoch and hence different magnitude of the vertical displacements. The respective RMSDs, SDs and biases are presented in Fig. 4.

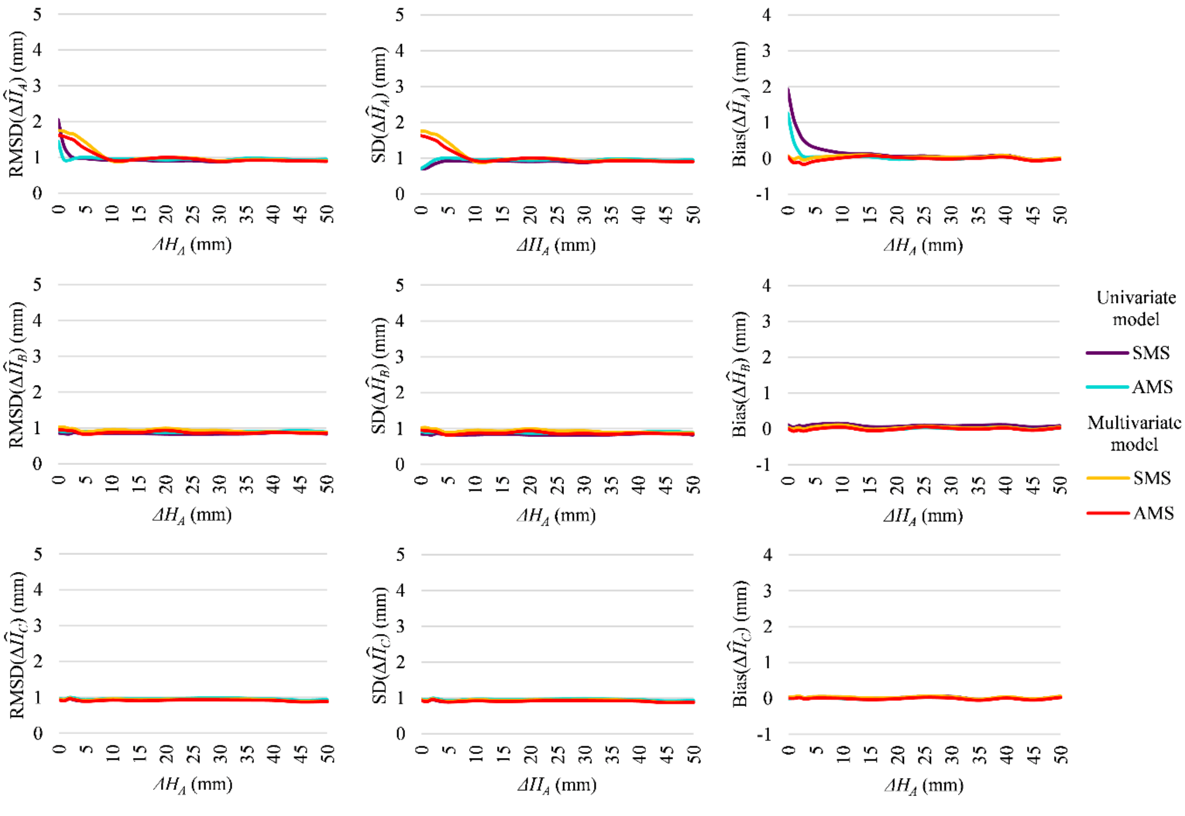

Fig. 4. RMSDs, SDs and biases of point displacements for different values of ΔH_A; where $\Delta H_B = 20$ mm, $\Delta H_C = 50$ mm (Variant III)

The accuracies and the biases of $\Delta\hat{H}_A$ achieve lower values for both SMS and AMS estimates for relatively small ΔH_A. Then $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ and $\mathrm{SD}\left(\Delta\hat{H}_A\right)$ are better for the univariate model whereas $\text{Bias}\left(\Delta\hat{H}_A\right)$ for the multivariate model. However, the results for all estimates considered are close to each other for bigger ΔH_A. It is very similar situation to that in Variant I. The estimate accuracy of the rest object point displacements are close to 1 mm and to each other for all ΔH_A considered. Furthermore $\text{Bias}\left(\Delta\hat{H}_B\right)$ and $\text{Bias}\left(\Delta\hat{H}_C\right)$ are neglectable. Generally, almost always the parameters are similar to each other for both approaches to M_{split} estimations and for both models considered. Considering Variants I and III, one can say that the accuracy and empirical systematic bias of M_{split} estimates depend on the point displacements for the univariate model as well as for the multivariate model.

3.4 Variant IV for $H_B^{II} = 20\text{ mm}$, $H_C^{II} = 50\text{ mm}$ and $h_1^{II} + 5\text{ mm}$

In last variant considered here the vertical displacements of the object points B and C are the same as in Variant III and additionally the observation h_1^{II} is affected by a gross error of magnitude 5 mm. Fig. 5 shows RMSDs, SDs and biases, respectively.

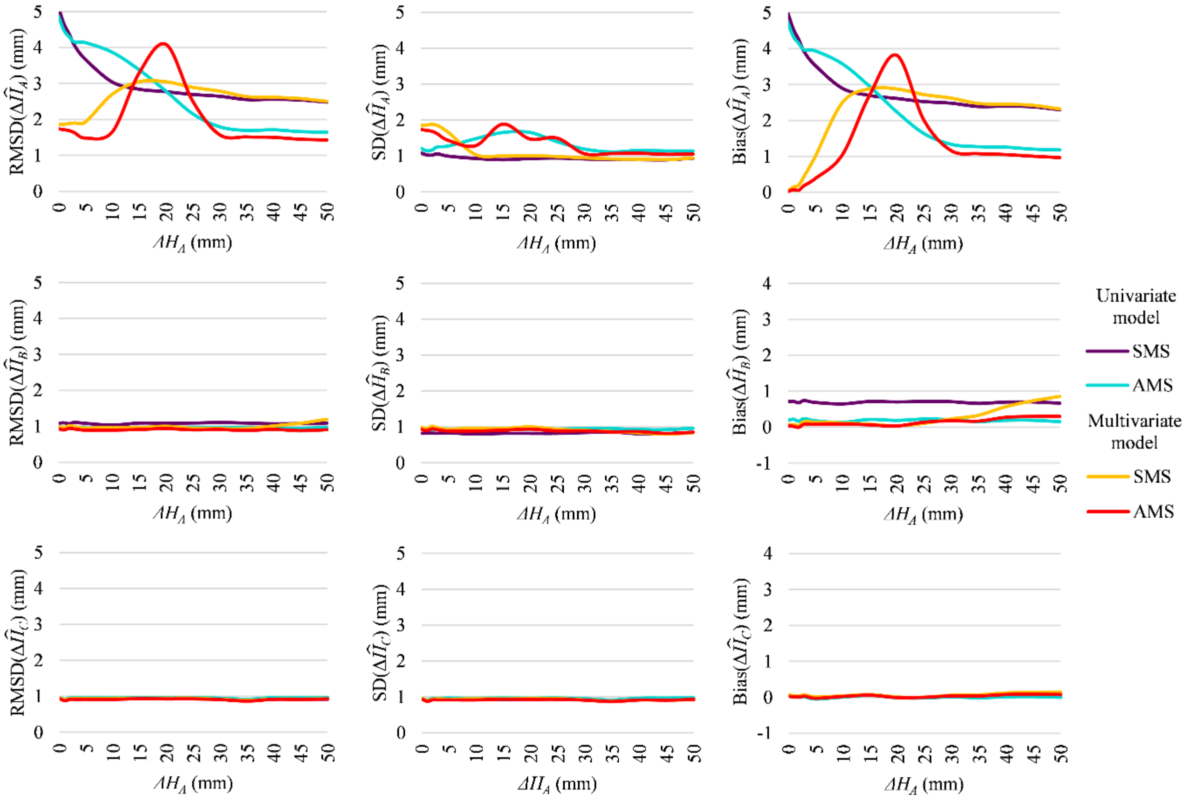

Fig. 5. RMSDs, SDs and biases of point displacements for different values of ΔH_A; where $\Delta H_B = 20$ mm, $\Delta H_C = 50$ mm and $h_1^{II} + 5\text{ mm}$ (Variant IV)

Considering $\text{RMSD}\left(\Delta\hat{H}_A\right)$, there are visible discrepancies between models used, especially for rather small values of ΔH_A. For growing values of ΔH_A, the differences between

$\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ which are obtained for different models are very small. In the context of the univariate model $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ are superior for AMS estimation for $\Delta H_A > \Delta H_B$. $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ for AMS estimation of the multivariate model are usually better than for SMS estimation. The exceptions are results obtained for ΔH_A close to ΔH_B. Then $\mathrm{RMSD}\left(\Delta\hat{H}_A\right)$ for AMS estimates are significantly bigger. That results from the coincidence between ΔH_A and ΔH_B. In the case of $\mathrm{SD}\left(\Delta\hat{H}_A\right)$ such a coincidence between ΔH_A and ΔH_B is also noticeable, even for the univariate model. What is more, the values of $\mathrm{RMSD}\left(\Delta\hat{H}_B\right)$, $\mathrm{RMSD}\left(\Delta\hat{H}_C\right)$ as well as $\mathrm{SD}\left(\Delta\hat{H}_B\right)$, $\mathrm{SD}\left(\Delta\hat{H}_C\right)$ are close to each other for all considered estimates. Note that the biggest biases are obtained for $\Delta\hat{H}_A$, which results from the location of the outlier. Such biases confirm conclusions from the analysis of RMSDs. For the object points *B* and *C*, the biases are neglectable for AMS estimates for both models considered. However, in the case of the object point *B* there are some discrepancies between models applied in SMS estimation. Variants II and IV confirm that occurrence of gross error in observation set might influence the accuracy and empirical systematic bias of M_{split} estimates for the univariate and multivariate models, especially the estimates of these object points which are closest to the outlier.

4. CONCLUSIONS

The paper presents an investigation of the univariate and multivariate models applied in deformation analysis based on two approaches to M_{split} estimation, namely the squared M_{split} estimation and the absolute M_{split} estimation. The results are obtained for four different variants. The empirical tests confirm that the accuracy of M_{split} estimates might depend on the values of the point displacements. Generally, if there is no gross error in the observation set, the univariate model in M_{split} estimation gives better accuracy. It is especially vivid for displacements which are close to 0 mm. This is very important from the practical point of view since small SDs result in smaller RMSDs. Thus, the application of the univariate model is especially advisable when such displacements are relatively small. On the other hand, systematic biases are smaller for the multivariate model. However, smaller biases not always result in better accuracy. When the observation set contains an outlier, there are visible significant differences between applications of the univariate or multivariate models. One can say that these two approaches simply supplement each other. Generally, the choice of the appropriate model considered in M_{split} estimation allows to obtain superior accuracy. The outcomes of executed empirical tests confirm that the accuracies (both RMSDs and SDs) and also empirical biases of both approaches to M_{split} estimates depend not only on the occurrence of gross errors in the observation set but also on the values of the point displacements. Finally, one can conclude that AMS method gives generally results which are less sensitive to gross error than SMS method does.

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BIOGRAPHICAL NOTES

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